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ABSTRACT

This text was written for junior high school teachers who wish to have more mathematical background on number systems. It is particularly useful for teachers who teach SMSG materials at grades 7 and 8. Chapters included are: (1) Introduction; (2) Numeration; (3) The Whole Numbers; (4) Divisibility and Properties of Whole Numbers; (5) The Non-Negative Numbers Rational; (6) Ratios, Decimals, and Applications; (7) Rational Numbers; (8) The Real Numbers; and (9) Equations and Graphs. The appendices include answers to problems and exercises and a selective bibliography. (RH)

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STUDIES IN MATHEMATICS

VOLUME XIV

Introduction to Number Systems

By Burton W. Jones

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STUDIES IN MATHEMATICS

VOLUME XIV

Introduction to
Number Systems

Edited by Burton W. Jones

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PREFACE

This book was written at the request of Professor E. G. Begle of the School Mathematics Study Group with the help of the following committee:

Professor Richard Dean, California Institute of Technology
Miss Muriel Mills, Hill Junior High School, Denver, Colorado
Professor G. Baley Price, The University of Kansas
Professor Warren Stenberg, The University of Minnesota

These persons not only assisted in planning the manuscript but twice during its development read it in detail and provided the author with many valuable suggestions and criticisms, most of which were incorporated in the revisions. He does wish to express his gratitude for their encouraging assistance and their thoughtful and fruitful ideas.

The author also wishes to acknowledge the assistance of Mrs. Arleen McClurg for the critical and detailed reading of the manuscript in its final stages.

Above all, thanks are due to Dr. Begle and the School Mathematics Study Group who, with the support of the National Science Foundation, made this writing possible.

Burton W. Jones

The University of Colorado
December, 1965

TABLE OF CONTENTS

PREFACE

Chapter 1. INTRODUCTION 1

Chapter 2. NUMERATION 5

2.1 Introduction 5
2.2 Development of numeration 6
2.3 The decimal system 10
2.4 Numeral systems to other bases 11
2.5 The duodecimal system 12
2.6 The numeral system to the base five 16
2.7 The numeral system to the base two 18

Chapter 3. THE WHOLE NUMBERS 23

3.1 Introduction 23
3.2 The set of whole numbers 24
3.3 Addition of whole numbers 28
3.4 Multiplication of whole numbers 32
3.5 The distributive property 34
3.6 Subtraction 37
3.7 Cancellation properties 40
3.8 Division 41
3.9 Inequalities 43
3.10 Looking backward 46

Chapter 4. DIVISIBILITY AND PROPERTIES OF WHOLE NUMBERS 49

4.1 Introduction 49
4.2 Divisibility 50
4.3 Prime numbers 52
4.4 Greatest common factor 55
4.5 Least common multiple 60
4.6 Tests for divisibility 64
4.7 Looking forward to the integers 68
4.8 Some properties of prime numbers 68
4.9 Other systems 71

Chapter 5. THE NON-NEGATIVE NUMBERS RATIONAL 75

5.1 Introduction 75
5.2 The definition of rational numbers 76
5.3 Equality of rational numbers 78
5.4 "Simplest" or "canonical" forms 81
5.5 Multiplication of rational numbers 84
5.6 Division of rational numbers 85
5.7 Addition of rational numbers 88
5.8 The basic structure of rational numbers 90
5.9 Subtraction of positive rational numbers 94
5.10 Density 95
5.11 Ordered number pairs 98

Chapter 6. RATIOS, DECIMALS AND APPLICATIONS 103

6.1 Introduction 103
6.2 Ratios 104
6.3 Decimals 104
6.4 Scientific notation and approximation 107
6.5 Decimal expansions 110
6.6 Percentage 113

Chapter 7. RATIONAL NUMBERS 117

7.1 Introduction 117
7.2 Definition of negative integers 117
7.3 Subtraction of integers 121
7.4 Multiplication of negative integers 122
7.5 Absolute value 124
7.6 The structure of the set of integers 126
7.7 Negative integers as number pairs 127
7.8 Negative rational numbers 130
7.9 Properties of the rational number system 130
7.10 The rational numbers as number pairs of integers 132
7.11 Inequalities 132
7.12 Betweenness 137
7.13 The triangle inequality 138

Chapter 8. THE REAL NUMBERS 141

8.1 Introduction 141
8.2 Definition of a real number 142

Chapter 9. EQUATIONS AND GRAPHS 151

9.1 Introduction 151
9.2 Solving equations 152
9.3 Solving inequalities 156
9.4 Why solve equations or inequalities? 161
9.5 Relations and functions 162
9.6 Applications to ordered pairs 167
9.7 Pairs of linear equations 168
9.8 Application 173

Appendix 1. SETS 179

Appendix 2. COUNTABLE SETS 183

Appendix 3. COMPLEX NUMBERS 187

Appendix 4. THE STRAIGHT LINE 199

Appendix 5. MODULAR ARITHMETIC 205

BIBLIOGRAPHY 211

ANSWERS TO PROBLEMS 215

INDEX

Chapter 1

INTRODUCTION

As its title suggests, this is a text written especially for those junior high school teachers who wish to have more mathematical background on number systems. Though, naturally, the author has in mind those using SMSG materials for grades 7 and 8, he also is writing for the larger audience of those who are interested in teaching youngsters arithmetic from a "modern" point of view.

What is this so-called "modern approach"? It is, in fact, as old as the hills, and every outstanding teacher in his or her time had a modern approach, from Socrates to Mark Hopkins to the most outstanding teacher you ever had. "Modern Mathematics" has come to be associated in the minds of many with words like: set, number systems to different bases, modular systems, commutative, inverse, etc. None of these words are new to mathematics; a course may be "modern" without using any of them, or it can use all of them without being "modern."

We have completed the circle to the question at the beginning of the previous paragraph. To most of the leaders of the movement for change, the now overworked term seems to mean fundamentally a certain point of view. It is to teach the student not only to manipulate but to know why and what for, to cultivate the inquiring mind and the love of inquiry, to develop the faculty of precise expression, to see and utilize relationships.

Why all this outcry for change? Some think it was due to Sputnik, but its beginnings antedated this achievement of Russia. It is that to live we must change. Things are moving so rapidly that we must be able to use machines, to be sure, but much more is needed--we must make new ones, and this requires a knowledge of how machines are made, what they can do, how one is related to the other. It is one of the paradoxes of the present age that in a contracting world, communication is fast becoming the prime problem; hence we must develop efficient means of dispensing and receiving information. This requires precise technical language.

In fact, the language, ideas and methods of mathematics are more and more pervading the life of the world. This is partly due to the enlargement of mathematics itself: such branches as linear programming, queuing theory, game theory are infants in the mathematical world, and even such a branch as topology is only a little past adolescence. There is much more to learn and so little time to learn it. But also the world is reaching toward mathematics and mathematical thought for help in solving (as well as creating) its problems. Though computing machines rule all our lives these days, the use of mathematics transcends the computational and, as machines take over the calculation, mathematics is freed for the realm of ideas. For instance, one controversy in theoretical physics revolves around the question: In certain fields will increased knowledge enable us to predict, or are certain phenomena unpredictable and only a matter of chance? This has its parallel in mathematics in the discovery that there are unprovable theorems; this does not mean theorems which have not been proved but ones which *cannot* be proven true *or* false. In this case mathematics has not provided the answer for the physicist, but it does seem clear that if he does find his answer, it will be by means of mathematics.

As was stated above, the aim of this book is to help the teacher acquire background for the material which is presented to the junior high school student in the books of the School Mathematics Study Group and other books with similar coverage. It is obvious that any teacher of any subject must know more than he teaches so that he will not be the slave of the textbook nor live in fear of his bright students. Hence we here deal with the development of the system of rational numbers beginning with the properties of the integers as we know them, why they are important, why we make certain definitions in extending the set of integers to the rational numbers and what are their consequences. Hand-in-hand go the graphical representation of numbers and pairs of numbers, for the interplay of geometry and number is important to both branches. Since the idea of equation is also fundamental to the understanding of various connections, we include something of this subject. The real number system, because of its complexity, is dealt with only very intuitively, and complex numbers are relegated to an appendix. Most of all is the attempt made to develop a mathematical structure of numbers based on reasonable axioms and developed along paths pointed out by intuition and made secure by proof. The reader is told where we are going and the reason for the particular path taken as well as possible alternate paths. Too often has dogma ruled in mathematics; here we try to make the point that the only authority is reason (and some experience) and that we are all humble before the shrine of learning.

Any civilization dies when the youth wish merely to preserve what they inherited from their fathers and fear the new. We must teach them what we know, but that is not enough. We must also teach them that they can extend their own knowledge far beyond where we can lead. By letting them find some relationships that we know, we can hope that they will see some which we do not know. If we can show them how discoveries are made, perhaps they can see the joy of discovery and so cultivate in themselves the power to make discoveries which are really their own.

It is somewhat in the spirit of the above that this book is written. By describing why certain things seem important to him, the author hopes to cultivate in the teacher the art of picking out what is important in view of his general aim; by demonstrating why certain processes are as they are, perhaps he can bring the teacher to the point where he can answer similar questions without assistance; by pointing out some relationships, perhaps he can show the teacher how to find relationships not mentioned here. The reader might like to compare some of this introduction with Chapters 1 through 4 of reference 1 in the Bibliography.

There are two kinds of exercises throughout the book. Those labelled "problems" are usually extensions and applications of the textual material and are definitely part of the development of the theory. Complete answers are given in the back of the book. The second kind, labelled "exercises," are generally of a more routine nature. Partial answers to some of these are given.

Chapter 2
NUMERATION

2.1 Introduction

In this chapter we are primarily concerned with the way numbers are and may be written. A little of the historical development of number sense and symbols is included to stress the idea that there are many ways of writing a number and also that various bases were and still are used. This leads naturally into a discussion of numeral systems to different bases and some of the consequences of the various notations. It is impossible to go very far in this direction without using the distributive property, $a(b + c) = ab + ac$, of numbers, as well as some of the other properties, and for this reason the teacher may find it better to change the order in her classroom. But the author feels that this fundamental property is familiar to his readers and that it is more important to keep the historical background close to the different numeral systems. Also it is important to stress that there are some properties which stem from the notation in which the number happens to be written and others which are properties of the numbers themselves. By this means we can separate these two sets of properties in Chapters II and III, though later in Chapters IV and VI both types of properties are dealt with.

Soon we come face to face with the distinction between "numeral," the notation, and "number", the abstract idea. This distinction can certainly be overemphasized, but it is important to have in mind. Consider

$$\text{three}\ ,\quad 3\ ,\quad 6 - 3\ ,\quad \frac{18}{6}\ ,\quad \text{III}\ .$$

These are all different ways of writing the number three. They are certainly not the same, but they represent the same number. This distinction is perhaps most strikingly exhibited in the difference between "numeral systems" (or "numeration") and "number systems." Whatever the words used, there are certainly two kinds. In this chapter is a discussion of numeral systems to various bases. These are just different ways of writing the set of integers; the numbers are the same for all. But also there are different number systems. The set of real numbers is different from the set of rational numbers. In

Appendix V is a brief discussion of a number system which has only twelve numbers in it and which is in some respects quite different from our familiar system. The distinction between number and numeral becomes important and difficult when one is dealing with fractions. We quite frequently use the same word for two different ideas outside of mathematics, and there is no harm in this provided that we know the difference in each particular situation. Many times we do not: for instance, there are two ways to answer the question "Who are you?". You can give your name, or you can tell something about yourself, and you must guess which the question means. So the suggestion of the writers is that the teacher carefully make the distinction between number and numeral where failure to do so would create confusion.

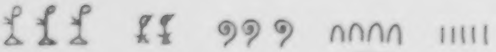
1.3 Development of numeration

Basic notion in the idea of number is that of a one-to-one correspondence. Though this term is a technical one, the idea of such a correspondence must be very ancient. To "count" the people in a primitive village, there was undoubtedly a system of making a notch in a stick or some other mark for each person. So each mark would correspond to a person and only one person. The number of marks would be the same as the number of persons. Each mark would be in a way a kind of map of a person. It might not have any of his characteristics (long, short, fat, thin), but it would merely stand for the person in the count. Actually we find it convenient even today in counting the votes in an election, for instance, to keep track by making a mark for each vote, though we usually introduce the slight refinement of grouping the marks perhaps by fives.

The next step after the tally marks would be to have a symbol different from the tally mark to represent a specified number of such marks. The most ancient records we have of such a procedure are of the Egyptians who, as far back as 3000 B.C., could express numbers up to millions. Their symbols were:

Our numeral	Egyptian symbol	Object represented
1	𓏺	stroke or vertical staff
10	𓎆	heel bone
100	𓍢	coiled rope or scroll
1000	𓆼	lotus flower
10,000	𓂭	pointed finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

So when they wrote a number, they just repeated the appropriate symbols the required number of times, and in no case, except for millions, would it be necessary to repeat a symbol more than nine times. For instance, 3,002,345 would be



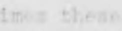
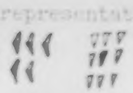
The Greeks and Hebrews used their respective alphabets for their numerals; that is, each of the numbers from one through nine, the tens through ninety, the hundreds through nine hundred had a letter of the alphabet associated with it. The Hebrews used the same symbol for 1000 as for 1, but the Greeks made the distinction by use of an extra symbol like the solidus: / . But for neither was there any sense of place value. It seems rather remarkable that for all their supremacy in geometry, the Greeks apparently did not progress far in a notation for numbers. However, certain facts about the numbers themselves were familiar to them. (Note our discussion in Chapter IV of the euclidean algorithm and the proof which was known in Euclid's time of the existence of an infinitude of prime numbers.)

The Roman numerals are used enough today so that they seem familiar:

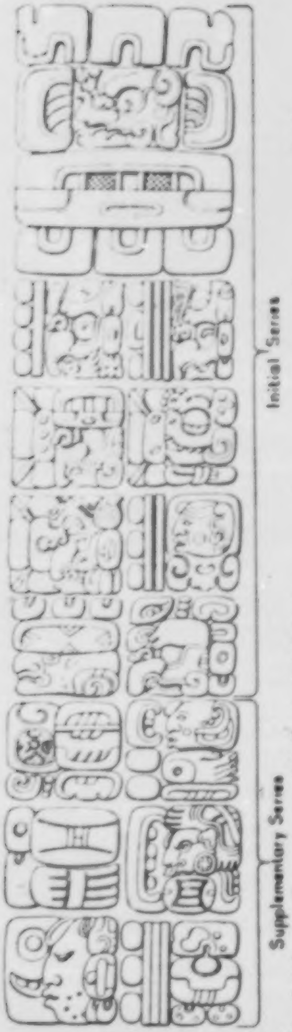
Our numeral	1	5	10	50	100	500	1000
Roman numeral	I	V	X	L	C	D	M

Again they indicated a number by writing the appropriate symbols a sufficient number of times, and in their case no symbol except M need be written more than four times. They introduced two modifications: first they wrote their symbols so that one symbol would be to the left of another symbol if the first represented a larger number and the two amounts were to be added, while if the symbol to the left is one of I, X, C and represents a smaller number, then it would be subtracted. For example, VI means six and IV means four; similarly CD would represent four hundred and DC six hundred, and XL is forty while LX is sixty.

But DM, LC, VX would not be written because simpler symbols for them would be D, L and V, respectively; similarly IIV would not appear. It is rather strange that the Romans progressed so little beyond the Egyptians in the representation of numbers.

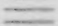
The Babylonians about 2000 B.C. seem to have been the first to introduce place value. Their symbol for 1 was a stylus stroke , and for ten a combination of two, like this . The numbers up through 59 were represented by repeating the required number of times these two symbols. For instance, their representation of our 34 was  , and their representation of 59 was  . So far there was nothing really new about their system. But for numbers greater than 59 they used place value, which was something radically new. For instance, 𒁹𒁹𒁹 𒁹𒁹 would represent 3 × sixty + 2 or 182 in our notation. Also 𒁹𒁹𒁹𒁹𒁹 𒁹𒁹𒁹 𒁹𒁹 would represent 5 × (sixty)^2 + 3 × (sixty) + 2 .

This use of position was a big step forward because with just their two symbols they would be able to represent a number of any size. Their chief lack was that they did not have a symbol for zero, and hence it would be difficult to tell whether 𒁹 𒁹 meant 2 or 1 × (sixty) + one or 1 × (sixty)^2 + 1.



Initial Series Introducing Glyph Grotesque head in center is the only variable element of this sign. This is the name glyph of the deity who is patron of the month (here Cumhu) in which the Initial Series terminal date falls	
9 *baktuns* (9 × 144,000 days = 1,296,000 days)	17 *katuns* (17 × 7,200 days = 122,400 days)
0 *tuns* (0 × 360 days = 0 days)	0 *uinals* (0 × 20 days = 0 days)
0 *kins* (0 × 1 day = 0 days)	13 Ahau (day reached by counting forward above total of days from starting point of Maya Era)
Glyph G9 Name glyph of the deity who is patron of the Ninth Day in the nine-day series (The Nine Gods of the Lower World)	**Glyph F** Meaning unknown
Glyphs E and D Glyphs denoting the moon age of the Initial Series terminal date, here "new moon"	**Glyph C** Glyph denoting position of current lunar month in lunar half-year period, here the 2d position
Glyph X1 Meaning unknown	**Glyph B** Meaning unknown
Glyph A9 Current lunar month, here 29 days in length. Last glyph of the Supplementary Series	18 Cumhu (month reached by counting forward above total of days from starting point of Maya Era). Last glyph of the Initial Series

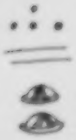
Examples of an Initial and a Supplementary Series: east side of Stela F, Quirigua.

The Mayans in Central America in the third or fourth century B.C. had both a place notation and a symbol for zero. They had fundamentally three symbols: one to four dots indicating units, a horizontal line* indicating five, and a shell which represented zero. They wrote their numerals vertically. Thus,  would represent 19 ($= 3 \times 5 + 4$). Our number twenty would be represented by .

They had names and symbols (called "glyphs") corresponding to our ten, hundred, thousand, etc., except that theirs were for powers of twenty (with one exception noted below) instead of powers of ten. Thus, with the exception noted below their names were:

Our number	1	20	20^2	20^3	20^4	20^5	20^6	20^7	20^8
Mayan name	kin	uinal	tun	katun	baktun	pictun	calabtun	kinchiltun	alautun

At least this was the system except when their reckoning had to do with the calendar. For the calendar one tun was eighteen uinals instead of twenty: from that point on it continued as above; that is, one katun was twenty tuns, one baktun twenty katuns, and so forth. With this alteration, their tun is equivalent to our 360, which is close to the number of days in the year. For example, consider the symbol:



If this had to do with the calendar, it would be:

$$0 + 0 \cdot 20 + 10(20 \cdot 18) + 3(20 \cdot 18 \cdot 20) + 1(20 \cdot 18 \cdot 20 \cdot 20) =$$
$$3600 + 21,600 + 144,000 = 169,200 .$$

If it had nothing to do with the calendar, it would be:

$$0 + 0 \cdot 20 + 10 \cdot 20^2 + 3 \cdot 20^3 + 1 \cdot 20^4 =$$
$$4,000 + 24,000 + 160,000 = 188,000 .$$

Thus the Mayans antedated by about a thousand years the Hindu introduction of place value and a symbol for zero. The Mayan's knowledge of the calendar and the movements of some of the heavenly bodies was remarkably accurate. In fact, in the sixth or seventh century of the Christian era, the ancient Mayan-astronomer priests at Copan had a calendar slightly more accurate than for Gregorian leap year correction. Our means of correction is to have an additional day every four years (leap years) except for centuries which are not multiples of 4: that is, 1700, 1800, 1900 were not leap years but 2000 will be. This gives a year of 365.2425 days. The comparative figures are:

---

*Sometimes the lines appear vertically.

Modern astronomy	365.2422
Julian year	365.2500
Present Gregorian year	365.2425
Mayan year	365.2420

The Mayan year had 18 months of 20 days each and one month of 5 (or 6) days at the end.

In 1939 the National Geographic Society - Smithsonian Institution discovered in Tabasco, Mexico (near Yucatan), remains of the lost Olmec civilization which flourished from 1500 to 800 B.C. preceding the Aztec. Among these remains was a stone slab bearing the oldest date found in the New World, corresponding to November 4, 291 B.C. The numerals on this slab were those of the Mayans, indicating that they were not the first to use their system of numeration.

The Hindu symbols which we use today moved from India to Arabia near the eighth century and did not arrive in Spain until the tenth century.[1] Here we have the familiar symbols for the numbers from one through nine, the symbol for zero, and the place value which seems to have begun with the Babylonians.

2.4 The decimal system

We have seen that the Babylonian, Mayan and Arabic systems all had symbols which had different meanings according to their place in the numeral. They depended on what might be called the unit for the place. For the Babylonians, powers of 60 were involved; for the Mayans, powers of 20; and for us, powers of ten. That is, for the Babylonians, if a , b , c were groups of symbols representing numbers between 1 and 59, a , b , c would mean $a(\text{sixty})^2 + b(\text{sixty}) + c$.

For the Mayans

$$\begin{matrix} a \\ b \\ c \end{matrix},$$

where again a, b, c are groups of symbols for numbers between 1 and 19 means $c + b\,(\text{twenty}) + a\,(\text{twenty})^2$ (if a calendar calculation is not involved).

For our notation we can write the symbols close together without spacing since only a single symbol is connected with each power of ten. Thus abc (not a product) would mean $a \times 10^2 + b \times 10 + c$.

The number which appears to the various powers is called "the base of the numeral system." Thus the Babylonians used a base of sixty: the Mayans, with one modification, a base of twenty: and we use one of ten.

We shall consider other systems in the sections following, but for us the most important one is the system to the base ten which we call the decimal system. We know that 12305 means

$$1 \cdot 10000 + 2 \cdot 1000 + 3 \cdot 100 + 0 \cdot 10 + 5 =$$

$$1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 0 \cdot 10 + 5 .$$

(In the second line we use the equivalent shortened notation where the small raised number, called the exponent, indicates how many times 10 occurs in the product. For instance, $10^4 = 10{,}000 = 10 \cdot 10 \cdot 10 \cdot 10$.) In the decimal system it is very easy to multiply by 10. Suppose we do it the long way, with the number above to see what is going on.

$$(12305) \cdot 10 = (1 \cdot 10^4 + 2 \cdot 10^3 + 3 \cdot 10^2 + 0 \cdot 10 + 5) \cdot 10 .$$

By virtue of the fact that $10^2 \cdot 10 = (10 \cdot 10) \cdot 10 = 10^3$, etc., we have, using the distributive property (see Chapter III),

$$(12305) \cdot 10 = 1 \cdot 10^5 + 2 \cdot 10^4 + 3 \cdot 10^3 + 0 \cdot 10^2 + 5 \cdot 10 = 123050 .$$

So we multiply by ten by adjoining a zero at the right side of the number. To multiply by 100 we would adjoin another zero, and so forth. It is equally easy to divide by 10, but we leave that to a later chapter.

2.4 Numeral systems to other bases

If we were to use our scheme of writing numbers for a numeral system to the base sixty, we would have to have a symbol for each of the numbers from 0 through 59, which would of course be very awkward. Even for a base twenty we would need ten more symbols than we now have. This would make our numerals more compact, but the increased number of symbols would mean that we would have a larger multiplication table to learn. To show what may happen with different numeral systems, we shall here consider in some detail three different ones. But the teacher is warned that the purpose of such consideration is not to give the students proficiency in computation in other bases but merely to illustrate by contrast what really goes on when we perform the familiar manipulations in our decimal system. Hence here is one place where the teacher should not try to cultivate in the students any great adeptness in the manipulations of the fundamental processes in numeral systems other than the decimal one. For instance, the properties of the multiplication table for the numeral system to the base twelve in the next section are interesting as well as their use in manipulations, but certainly there is no use in memorizing the table or drilling for rapidity of calculation. To substitute mechanics in the duodecimal system for that in the decimal system would be fruitless.

2.5 The duodecimal system

This is the system to the base twelve. Here we need two new symbols, and we might as well use the corresponding letters: using the symbol t for ten and e for eleven. Thus we have the symbols:

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e.$$

Thus, e230t means

$$\text{eleven (twelve)}^4 + 2(\text{twelve})^3 + 3(\text{twelve})^2 + 0(\text{twelve}) + \text{ten}$$

which, in the decimal system, is:

$$11(12)^4 + 2(12)^3 + 3(12)^2 + 0(12) + 10 =$$
$$= 228{,}096 + 3{,}456 + 432 + 10 = 231{,}994 .$$

First consider the process of addition, for instance, for the sum 9 + t in the duodecimal system. One method for computing this would be to note that nine plus ten in the decimal system is nineteen which, in turn, is seven more than twelve. Thus in the duodecimal system 9 + t = 17 . This is a process of converting to the decimal system, performing the addition, and converting back. A second method is perhaps a little more efficient, especially if you are used to using it for the decimal system. For this we notice that 9 lacks 3 of twelve, and we think 9 + t = 9 + 3 + (t - 3) = twelve + 7 = 17 .

The process of adding a column of numerals in the duodecimal system is the same as for the decimal system. For instance, for the sum

$$\begin{array}{r} 57 \\ 8e \\ t9 \\ \hline 23 \\ 1e \\ \hline 213 \end{array} ,$$

we add the right column, add the left, and keeping our columns in line, add the results. Written out this would be

$$57 + 8e + t9 = 5(\text{twelve}) + 7 + 8(\text{twelve}) + e + t\cdot(\text{twelve}) + 9$$
$$= (5 + 8 + t)\cdot(\text{twelve}) + (7 + e + 9) ,$$

using the commutative, associative, and distributive properties. (For a description of these properties see Chapter III.) Then $(5 + 8 + t)\,(\text{twelve}) = (1e)\,(\text{twelve}) = (\text{twelve})^2 + e(\text{twelve})$. So our sum is equal to

$$(\text{twelve})^2 + e(\text{twelve}) + 2(\text{twelve}) + 3 .$$

Use the distributive property again on the two middle terms, and we have

$$(\text{twelve})^2 + 11(\text{twelve}) + 3 = 2(\text{twelve})^2 + (\text{twelve}) + 3 ,$$

which is written 213. When we write it out this way, two things are accomplished: we see what the process is, and we appreciate more, perhaps, how simply the mechanical process gives us our result.

When it comes to multiplication, we could again use the "convert, multiply, and convert back" technique, but here it is probably simpler to construct and use a multiplication table.

So the multiplication table for the duodecimal system is:

	1	2	3	4	5	6	7	8	9	t	e
1	1	2	3	4	5	6	7	8	9	t	e
2	2	4	6	8	t	10	12	14	16	18	1t
3	3	6	9	10	13	16	19	20	23	26	29
4	4	8	10	14	18	20	24	28	30	34	38
5	5	t	13	18	21	26	2e	34	39	42	47
6	6	10	16	20	26	30	36	40	46	50	56
7	7	12	19	24	2e	36	41	48	53	5t	65
8	8	14	20	28	34	40	48	54	60	68	74
9	9	16	23	30	39	46	53	60	69	76	83
t	t	18	26	34	42	50	5t	68	76	84	92
e	e	1t	29	38	47	56	65	74	83	92	t1

As a matter of fact, the construction of the multiplication table is instructive. One can follow the pattern of successive addition, but for some of the rows an easier pattern emerges. Watch the succession of last digits in the multiples of 2, 3, 4, 6, 8, 9, t, e. Notice the symmetry of the table with respect to the diagonal from upper left to lower right. There is even a kind of symmetry for the other diagonal.

Using this table, let us find mechanically first of all the value of the following product in the duodecimal system, that is, using by analogy for this system the same process which we know so well in the decimal system.

$$\begin{array}{r} 7t6 \\ e2 \\ \hline 1\,3\,{}^{1}9\,{}^{1}0 \\ 7\,2\,{}^{9}7\,{}^{5}6 \\ \hline 7\,3\,e\,{}^{1}3\,0\,. \end{array}$$

Consider in detail the process of multiplying e by 7t6. From the table e · 6 = 56 , and hence we write the 6 and "carry the 5"; that is, add 5 to the next product. Thus we have next

$$5 + t \cdot e = 5 + 92 = 97\,.$$

Then we write 7 and add 9 to the next product. Thus we have 9 + 7 · e = 9 + 65, using the table. Here 9 + 5 = twelve + 2 = 12 , and thus

$$9 + 65 = 60 + 9 + 5 = 60 + 12 = 72\,.$$

To see what is back of the mechanical process, we can write it as follows:

$$\begin{aligned}(7t6) \cdot e2 &= e(\text{twelve}) \cdot (7t6) + 2 \cdot (7t6) = \\ &= e \cdot (7t6)\,(\text{twelve}) + 2 \cdot (7t6)\,,\end{aligned}$$

using the distributive property and the associative and commutative properties of multiplication. Then

$$\begin{aligned} e \cdot (7t6) &= e \cdot 7(\text{twelve})^2 + e \cdot t(\text{twelve}) + e \cdot 6 = \\ &= 65(\text{twelve})^2 + 92(\text{twelve}) + 56 \\ &= [6(\text{twelve}) + 5]\,(\text{twelve})^2 + [9(\text{twelve}) + 2]\,(\text{twelve}) + 5(\text{twelve}) + 6 \\ &= 6(\text{twelve})^3 + (5 + 9)\,(\text{twelve})^2 + (2 + 5)\,(\text{twelve}) + 6\,.\end{aligned}$$

Then 5 + 9 = 12, and so

$$(5 + 9) \cdot (\text{twelve})^2 = 12 \cdot (\text{twelve})^2 = (\text{twelve})^3 + 2(\text{twelve})^2\,.$$

Thus finally we have

$$e \cdot 7t6 = (6 + 1) \cdot (\text{twelve})^3 + 2 \cdot (\text{twelve})^2 + 7 \cdot (\text{twelve}) + 6 = 7276\,.$$

Writing it out this way soon becomes rather boring, but it does again impress us with the advantages of the mechanical system.

Now let us check the result by converting into and out of the decimal system. First

$$7t6 = 7 \cdot (\text{twelve})^2 + t \cdot (\text{twelve}) + 6\,,$$

which, in the decimal system, is

$$7 \cdot 144 + 10 \cdot 12 + 6 = 1134 .$$

Similarly,

$$e2 = e \cdot (\text{twelve}) + 2 ,$$

which in the decimal system is 134. Then the product is 151,956.

We could complete the check by converting 73e30 into the decimal system. But, to illustrate the conversion in the other direction, we elect to convert 151,956 into the duodecimal system. Here we need to write this number in the form:

$$* (\text{twelve})^3 + * (\text{twelve})^2 + * (\text{twelve}) + *$$

where the stars stand for unknown numerals from 0 to e, inclusive. We may have to begin with a higher power of twelve. Let us form a little table:

n	1	2	3	4
$(\text{twelve})^n$ in the decimal system	12	144	1728	20736

The fifth power would be in the neighborhood of 240,000, which is too large. The highest multiple of 20,736 less than 151,956 is 7, and we have

$$151{,}956 = 7 \cdot (20{,}736) + 6804.$$

The highest multiple of 1728 less than 6804 is 3, and we have

$$6804 = 3 \cdot 1728 + 1620 .$$

Also 1620 = 11(144) + 36, where

$$36 = 3 \cdot 12 .$$

Combining these results, we have

$$\begin{aligned} 151{,}492 &= 7(\text{twelve})^4 + 3(\text{twelve})^3 + e(\text{twelve})^2 + 3(\text{twelve}) + 0 \\ &= 73e30 \text{ in the duodecimal system.} \end{aligned}$$

This checks with our direct computation.

There is another method of conversion which is simpler mechanically but is a little harder to justify. First we show the computation, which consists in a sequence of divisions by twelve, and the recording of the quotients and remainders.

Quotients	Remainders
151,956	
12,663	0
1,055	3
87	e
7	3
0	7

Hence 151,956 in the decimal system is equivalent to 73e30 in the duodecimal system. Each line is obtained from the previous one by dividing by twelve.

Why does this work? To explain this, we look at a more general expression.

$$N = a(\text{twelve})^4 + b(\text{twelve})^3 + c(\text{twelve})^2 + d(\text{twelve}) + f \ .$$

If N is divided by twelve, the quotient is

$$N' = a(\text{twelve})^3 + b(\text{twelve})^2 + c(\text{twelve}) + d$$

with the remainder f. So the first remainder is the last digit in the numeral in the duodecimal system. Next d is the remainder when N' is divided by twelve and the quotient is

$$a(\text{twelve})^2 + b(\text{twelve}) + c \ .$$

So the process continues.

The usual process for division could be carried out also in this system, but it seems scarcely worth the effort. There seems to be no historical record of a race using consistently the duodecimal system, though it appears often in our own civilization: there are twelve inches in a foot; eggs are sold by the dozen; and in our calendar there are twelve months in the year. Twelve is much more satisfactory in these cases, since it is divisible evenly by 2, 3, 4, 6, whereas 10 is divisible only by 2 and 5 beside itself and 1. But the advantages of a change do not seem worth the trouble to most people, and the Duodecimal Society does not make much progress in converting the world.

2.6 The numeral system to the base five.

(The reader may prefer to omit this section.)

Much simpler than the duodecimal system is that to the base five. Here we have only five symbols, and the multiplication table is much simpler than our own. In fact it would not be much of an effort to memorize, if it were worth doing, which it is not. But here it is:

	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

Here we illustrate the process of multiplication by an example. The basis for the process, using the various properties of the number system, is the same as for the duodecimal system, or indeed any other of this type.

$$\begin{array}{r} 423 \\ 12 \\ \hline 1401 \\ 423 \\ \hline 11131 \end{array}$$

To illustrate the process of conversion in both directions, we can check the process as follows:

$$423 = 4(\text{five})^2 + 2(\text{five}) + 3 = 113 \text{ in the decimal system}$$
$$12 = 1(\text{five}) + 2 = 7 \text{ in the decimal system .}$$

The product in the decimal system is 791. To find what this number is in the system to the base five, we use the shortened form illustrated above for the duodecimal system:

Quotient	Remainder
791	
158	1
31	3
6	1
1	1
0	1

In each case we divide by 5 and write the quotient and remainder. Thus if we divide 791 by 5, the quotient is 158 and the remainder is 1; if we divide 158 by 5, the quotient is 31 and the remainder 3, etc. Hence 791 = 11131 in the numeral system to the base five.

There seems to be little use systematically of the numeral system to the base five, though the Roman numerals show this tendency and the Mayans had a symbol for five. The five also appears in the abacus.

2.7 The numeral system to the base two

The simplest of the numeral systems to different bases is that to the base two. Here the addition and multiplication tables are about as simple as could be imagined. Here they are:

+	0	1
0	0	1
1	1	10

×	0	1
0	0	0
1	0	1

Addition becomes very simple:

$$\begin{array}{r}
1011 \\
1101 \\
111 \\
\hline
11 \\
10 \\
10 \\
10 \\
\hline
11111
\end{array}$$

For the right column we have 1 + 1 = 10 and 10 + 1 = 11. Similarly for the other columns. Multiplication is also easy:

$$\begin{array}{r}
1011 \\
1101 \\
\hline
1011 \\
1011 \\
1011 \\
\hline
10001111
\end{array}$$

The conversions are as follows:

$1011 = 1(\text{two})^3 + 1(\text{two}) + 1 = 11$ in the decimal system

$1101 = 1(\text{two})^3 + 1(\text{two})^2 + 1 = 13$ in the decimal system.

The product is 143. To convert this to the base two, we have by successive divisions by two the following:

Quotient	Remainder
143	
71	1
35	1
17	1
8	1
4	0
2	0
1	0
0	1

Thus writing the numbers from left to right instead of from the bottom up, we have: 10001111 .

There are a number of applications of the numeral system to the base two. Fundamentally, the use of this system in computation is that there are only two symbols used and they can be made to correspond to the two positions of an electric switch. Thus a number could be represented by closed or open switches according to the occurrence of a one or a zero in corresponding places in its binary representation. It is for this reason that the binary system lies at the basis of the computing machine.

We show two applications of a puzzle nature. The first is the so-called Russian peasant method of multiplication. We illustrate it for finding the product of 573 and 25. In the left column we multiply each time by two and in the right we divide by two, discarding remainders:

573	25*
1146	12
2292	6
4584	3*
9168	1*

To get the product, we add the numbers on the left corresponding to the starred (that is, odd) numbers on the right, that is 573 + 4584 + 9168 = 14,325. The reason for this is seen if we add two columns--on the right the remainders and on the left the multiples:

1	573	25	1
2	1146	12	0
4	2292	6	0
8	4584	3	1
16	9168	1	1

The right column gives the digits in the representation of 25 in the binary system. That is, 25 is 11001 in the binary system; in other words

$$25 = 1 + 0 \cdot 2 + 0 \cdot 4 + 1 \cdot 8 + 1 \cdot 16 .$$

So the product of 25 by 573 is

$$(1 + 0 \cdot 2 + 0 \cdot 4 + 1 \cdot 8 + 1 \cdot 16)\, 573 =$$
$$1 \cdot 573 + 8 \cdot 573 + 16 \cdot 573 = 573 + 4584 + 9168 = 14{,}325 .$$

A second application is in the construction of a set of cards for a trick. Here is the set of numbers on each of them for the determination of numbers from 1 through 15.

1	3	5	7
9	11	13	15

2	3	6	7
10	11	14	15

4	5	6	7
12	13	14	15

8	9	10	11
12	13	14	15

The trick is this. You ask someone to select a number from 1 to 15, inclusive, and then pick out the cards on which this number lies. You then add the first numbers on the cards chosen and recover the number which was thought of. The reason for the success of this trick is that on the first card occur the numbers whose last digit in the binary system is 1, that is, the odd numbers. On the second card are the numbers whose next to the last digit in the binary system is 1. The third card contains those whose third digit counting from the right is 1, and the fourth card, those whose fourth digit counting from the right is 1. Thus, for instance, if the number is 13, its representation in the binary system is 1101 which corresponds to 13 found on all the cards except the second. That is, $13 = 1 + 0 \cdot 2 + 1 \cdot 4 + 1 \cdot 8 = 1 + 4 + 8$.

Another interesting application is to the game of Nim, which may be found in various references.

Exercises

1. "Four score and seven years ago" indicates a numeral system to what base? Find evidences in our civilization of use of the following bases: five, twelve, one hundred.

2. Arrange the following in order of size:

$$3^4 , 4^3 , 2^7 , 7^2$$

3. Perform the following calculations in the numeral system to the base twelve and check your results:

   a) eee · 35 b) tt + ele + 999

   c) eee · ee

4. Write the multiplication table of the numbers from 1 to 7 in the numeral system to the base eight.

5. In the numeral system to the base two, perform the following calculations and check your results:

   a. 11111 · 1111 b. 1010101 · 10101

   c. $\frac{1}{11}$ d. $\frac{1}{111}$

   (In the last two note that in this system, .1 means one-half, .01 one-fourth, etc.)

6. Can you draw any general conclusions from the examples in the previous exercise?

7. Write in the numeral system to the base seven the numeral for six dozen. What numeral systems do you use in the solution to this problem?

8. Is 11 in the numeral system to the base seven divisible by two or not? How would you test a number written in the numeral system to the base seven for divisibility by two?

Problems

1. Construct a set of five cards which can be used in the trick above to determine any number from 1 through 31.

2. An object is weighed by a balance on one side of which the object is laid and on the other are put certain specified weights. What weights would you use as a minimum set to weigh all objects of a whole number of ounces from 1 to 15, inclusive?

3. In the markets of Guatemala and other Central American countries, to weigh out the corn and other commodities there is a little kind of brass cup holding nested weights. The innermost weight is one-half an ounce, approximately; this with the first cup weighs one ounce; the second cup weighs the same as the first two weights together, that is, one ounce; the third weighs the same as the first three, and so on. If there are five cups in addition to the innermost weight, what amounts can be weighed on a balance with this series of weights?

References

(The underlined numbers refer to the Bibliography at the back of the book)

1 (Chapter 5), 2, 5 (Chapters 1 and 2), 8 (Chapter 3), 18, 19, 20 (Chapter 1), and 22 (Chapter 4).

Chapter 3

THE WHOLE NUMBERS

3.1 Introduction

The set of numbers

$$1, 2, 3, 4, \ldots$$

is called the set of natural numbers or of counting numbers. We shall usually use the latter term. This set with zero added we call the set of whole numbers. The reader should be warned that this terminology is not universal.

In this chapter we shall be concerned with properties of these numbers expressed in the decimal system, which are independent of what numeral system is used to represent them, that is, properties of the numbers themselves, irrespective of what notation is used to represent them. It is important to study these properties in detail because if we feel at home with numbers at all, we feel so with these numbers; their properties are familiar to us though perhaps we do not often think about them. Here there is no thought of trying to lay a firm foundation for a number system complete in all detail, but rather to become thoroughly familiar with these properties so that as we proceed to extend our number system, we will be at home in perhaps less familiar surroundings. Really what happens is that in each stage of extension of our number system and in the algebraic processes which follow, we make sure to manage it so that as many as possible of the familiar properties of the whole numbers carry over into the extension. And furthermore, the basis for the manipulative "rules" of algebra is again these fundamental properties of the whole numbers.

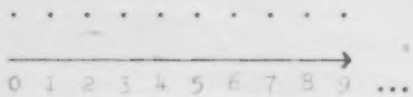
Not only is the study of these properties important for the forward view toward other numbers and algebra, but it is important looking backward over the common manipulative processes of addition, subtraction, multiplication, and division. The junior high school student presumably is familiar with these, but he is old enough to have the right to know why and, incidentally, get more practice in the processes in this new setting. (Notice also the last section of this chapter.)

Thus this chapter, rather than the previous one, is the place for thoroughness; for if the properties are understood in this familiar setting, it will be much easier to understand them in less familiar surroundings, for these same properties recur at each stage of the development. And yet it is true here, as in all branches of learning, that the now is never thoroughly understood until the hereafter makes it history.

3.2 The set of whole numbers

We saw in the previous chapter that the set of counting numbers is a kind of yardstick by which we can answer questions about how many. (See Appendix I for a short discussion of sets and the accompanying terminology.) If we want to compare the number of elements (or objects) in two sets, we can set up a one-to-one correspondence between them as far as possible and see if there are elements left over in either. Another method is to count the number of elements in both and compare the numbers, that is, exhibit a one-to-one correspondence between each of the given sets and a set of numerals. The set of counting numbers is useful for this purpose because there is always a "next one," and in virtue of this order we can tell at a glance (at least in our decimal system) which of two numbers is greater and hence which of the corresponding sets contains more elements, unless they have the same number of elements. It does not matter in what way we set up the one-to-one correspondence between the elements of a set and natural numbers, which we call counting, but it is vital that the numbers with which we count have order.

An excellent way to exemplify this is to set up points on a line (more accurately, a ray with end-point 0) in one-to-one correspondence with the whole numbers, using the geometric order from left to right to correspond to the numerical order as follows:



Just as we say that 13 is greater than 9 because 13 follows 9 in the whole number sequence, so on the number line the point corresponding to 13 is to the right of the point corresponding to 9. We designate this relationship by $13 > 9$ or $9 < 13$. In general, if a whole number occurs after another in the ordered sequence of whole numbers, we say that the former is greater than the latter or that the latter is less than the former. The symbol for "greater than" is $>$ and that for "less than" is $<$.

The two fundamental *properties of order* are:

1. If b and c are any two whole numbers, then exactly one of the following holds:

$$b > c, \; b = c, \; c > b \,.$$

2. If $a > b$ and $b > c$, then $a > c$ (the transitive property). These properties occur in other settings in mathematics and outside of the subject and are characteristic of what is often called an "ordered set."

More formally, we call a set S an *ordered set* with respect to a relationship R if it has the following properties:

1. If a and b are two different elements of the set, exactly one of the following holds:

$$a \, R \, b \,, \quad a = b \,, \quad b \, R \, a \,.$$

2. If $a \, R \, b$ and $b \, R \, c$, then $a \, R \, c$ (the transitive property). In the case of whole numbers above, R is $>$. Of course the same properties would hold if R meant $<$.

Another example of an ordered set would be a set of foods where $a \, R \, b$ means "I like food a better than food b," and the equality would mean that I like them equally well. This satisfies the two conditions in general, though for some inconsistent people the second property may not always hold.

If R signifies the relationship "is contained in" for pairs of sets, it is not, in general, an order relationship. To see this, consider the sets:

$$A = (r, s, t) \text{ and } B = (c, d, e).$$

Neither set of three letters is contained in the other nor are they equal.

A consequence of the order relationship is that of *betweenness*. A number is said to be between two others if it is less than one *and* greater than the other. In terms of our general relationship R, we can define betweenness as follows:

Definition: A set S is said to have the property of *betweenness* with respect to a relationship R if, for every three distinct elements a, b, c of S, at least one of the following holds:

$$a \, R \, b \, R \, c, \; c \, R \, b \, R \, a; \; b \, R \, a \, R \, c, \; c \, R \, a \, R \, b; \; a \, R \, c \, R \, b, \; b \, R \, c \, R \, a,$$

where $a \, R \, b \, R \, c$ means $a \, R \, b$ *and* $b \, R \, c$ but does *not* mean $a \, R \, c$, etc. In the first two cases b is said to be between a and c; in the third and fourth a is said to be between b and c; and in the last two c is said to be between b and a. Also " a is between b and c " is equivalent to " a is between c and b."

If (and this is the case we are concerned with) the set S is the set of points on a horizontal line, the relationship $a \; R \; b$ could mean: point a is to the left of point b on the line. Since the points are ordered on the line, the property of betweenness holds. In fact, it is not hard to see that this property follows from the property of being an ordered set. Let us show why this is so in terms of numbers and inequality. The proof for the general relationship would go in the same way. To be definite, we state a theorem:

Theorem: If any set of numbers is an ordered set for the relationship $>$, then it has the property of betweenness for this same relationship.

Proof: Suppose r, s, and t are three different numbers of an ordered set. Then there are four possibilities for s in relation to the other two:

i) $s > r$ and $s > t$,

ii) $s > r$ and $t > s$, when s is between r and t,

iii) $r > s$ and $s > t$, when s is between r and t,

iv) $r > s$ and $t > s$.

In case i), if $t > r$, then t is between r and s, while if $r > t$, then r is between s and t. In case iv) the situation is similar. This completes the proof.

But it is possible to have the property of betweenness, as we have stated it, without the properties of an ordered set. For instance, let A, B, C be the three vertices of an equilateral triangle, and let A R B mean "we can move point A into point B by a rotation of the triangle through an angle of 120° in the clockwise direction about its center" as indicated in the figure.

Then it is easy to see that

$$A \; R \; B \; R \; C, \quad B \; R \; C \; R \; A, \quad C \; R \; A \; R \; B,$$

and so in this sense each point is between the other two. The first property of order holds, since

$$A \; R \; B, \quad B \; R \; C, \text{ and } C \; R \; A,$$

but the second does not hold since A R B and B R C do not imply A R C; however, C R A does hold. The reader should be warned at this point that often the idea of betweenness is expanded to the point where it is equivalent to the property of order.

To return to the set of whole numbers, we know that it forms an ordered set for the relationship > and hence has the property of betweenness; that is, given any three whole numbers, exactly one is between the other two. But it is not true that given any pair of whole numbers, there is always one between them. In other words, the set of whole numbers is not dense. Let us define this notion precisely:

Definition: A set S with a relation R having the property of betweenness is called dense if, for any two elements of the set, there is a third element between them. (For another definition of density and more complete discussion, see Section 5.10.)

There is no whole number between 2 and 3 or, in fact, between any two whole numbers whose difference is 1. You may notice with surprise that the set of three vertices of a triangle given in the illustration above does have the property of denseness though, in an intuitive sense, it is anything but dense. More to the point is the fact which we can establish later that the rational numbers form a dense set, since if a and b are any two rational numbers, $\frac{(a + b)}{2}$ is a rational number which is between them.

To the author it does not seem advisable that junior high school students study these properties formally, but the knowledge of them should gradually evolve out of experience, and the good teacher can see to it that experience of this nature is had.

The numeral zero, as we saw in the previous chapter, has a most important role in the decimal system of numeration. But the number zero has more important properties peculiar to itself. In counting, it indicates the absence of what is counted; for instance, zero is the number of persons over twenty feet tall. Every whole number has a predecessor except zero. This is exhibited in a "count down": 5, 4, 3, 2, 1, 0; at the number zero the rocket is fired. Zero has a very special role, as we shall see, with respect to addition and multiplication.

Problems

1. Complete the fourth possibility in the discussion of betweenness above.

2. If, in an ordered set, a is between b and c, and c is between a and d, prove that a and c are between b and d.

3. Let A, B, C, D be the four vertices in clockwise order of a square and A R B mean "we can move from A to B by a rotation of the square through an angle of 90° or 180° in a clockwise direction about its center" with a similar meaning for any other pair of vertices. Is this set of points an ordered set? Does it have the property of betweenness? Is it dense? Explain.

3.3 Addition of whole numbers

The fundamental notion of addition for whole numbers is, as we know, associated with the idea of counting in the following way: if two sets of objects are distinct (that is, have no elements in common), then the number of objects in both sets together is defined to be the sum of the numbers of objects in the sets separately. For example, if A is the set (2, 3, 5, Δ, Σ) and B the set (r, s, t), A has five elements, B three, all different from those of A, and hence the combined set, which we call the union, has 5 + 3 or 8 elements. In notation this could be:

$$n(A) + n(B) = n(A \cup B), \text{ if} \tag{1}$$

A and B have no elements in common, that is, if their intersection is the null set and n(A) denotes the number of elements in A.

The first property of whole numbers which follows from this is that of closure: The sum of two whole numbers is a whole number.

A second property of addition which immediately follows from this relationship is a direct result of the fact that the combined set of A and B is the same as the combined set of B and A; that is, the union of sets A and B is the same as the union of sets B and A. In notation this may be written:

$$n(A) + n(B) = n(A \cup B) = n(B \cup A) = n(B) + (A). \tag{2}$$

The first equality follows from the definition, the second from the equality of $A \cup B$ and $B \cup A$, and the third again from the definition with B and A interchanged. This would hold for any pair of sets without a common element. So we have

The Commutative Property of Addition: If b and c are any two whole numbers, then

$$b + c = c + b \, .$$

In thinking of this, we perhaps assumed that neither of the letters stood for the number zero, but it also holds for zero because $n(A) = 0$ would mean that A is the null set and in that case $(A \cup B) = B$, and equations (2) above would give

$$0 + n(B) = n(B) + 0 \, .$$

Actually we have a little more--our first property of zero:

$$0 + c = c + 0 = c \, .$$

Zero is called the identity (or neutral) element for addition.

Of course this is not the way one would first teach these facts to a junior high school class, but, in the opinion of the author, the above is what should be in the teacher's mind to guide him or her in the presentation. Moreover, this approach might be useful for a review. The commutative property is part of the justification for adding two columns of figures first in one order and then in the other. It can be shown graphically by considering two rows of dots:

b dots
c dots

If we count the dots from left to right, we have $b + c$ dots; if from right to left, we have $c + b$ dots. Since the number of dots in both sets is independent of the order in which we count them, we have shown by this device that addition is commutative.

Suppose now that we have three sets A, B, and C. If we want to find the union of all three, that is, the set consisting of all the elements together, it would make no difference whether we combined A and B first and then this union with C, or A with the union of B and C. In notation

$$(A \cup B) \cup C = A \cup (B \cup C) \, ,$$

and since this is true, we could just as well write them both in the form

$$A \cup B \cup C \, .$$

This is the associative property for the union of sets. For example,

If $A = (5, 7, 6)$ $B = (=, [illegible])$ $C = (\Delta, Z, 2, 4)$

$A \cup B = (5, 7, 6, =, [illegible])$, $(A \cup B) \cup C = (5, 7, 6, =, [illegible], \Delta, Z, 2, 4)$.

$B \cup C = (=, [illegible], \Delta, Z, 2, 4)$, $A \cup (B \cup C) = (5, 7, 6, =, [illegible], \Delta, Z, 2, 4)$.

Perhaps this is simpler without all this notation: we are merely saying that if you wish to combine three sets, it makes no difference in what order you combine them.

Now, to make the connection with numbers, we assume that no pair of A, B, C has common elements and see that the numbers corresponding to the sets have the same property that the sets have; that is,

$$(a + b) + c = a + (b + c).$$

This is the associative property for addition of whole numbers. Notice that the order of the letters is the same on both sides of the equation, but the way in which they are combined is different. Just as for sets, this property allows us to write

$$a + b + c$$

without parentheses and without ambiguity.

This is a vital property for addition, since we can only really add two numbers at a time. So to find the sum

$$5 + 7 + 8 + 9 ,$$

for instance, we can first add 5 and 7, then that sum to 8 and that sum to 9. Each time we add just two numbers. The property of associativity and commutativity assures us that if we had worked from the right side to the left, the result would have been the same.

Finally, there is another property which is quite evident at this stage but is not so apparent in less familiar settings. In fact it is the basis for one of the methods of manipulation for the solution of equations. It is this:

If a and b are two names for the same whole number, then $a + c$ and $b + c$ are two names for the same number. In notation

$$a = b \text{ implies } a + c = b + c .$$

This is sometimes called the well-defined property. The reason for the name is this rephrasing of the property:

> The sum of two numbers is independent of the particular numerals used.

For instance, since $\frac{6}{3} = \frac{10}{5}$, then $\frac{6}{3} + 7 = \frac{10}{5} + 7$.

Exercises

1. In a booklet the last three digits of the numbers of travelers' checks run from 168 to 192, inclusive. How many checks does the booklet contain?

2. In each case below, find a number which may be used in place of x to make the equation true. Give reasons for your answer.

   a) $3 + 7 = 7 + x$ b) $(6 + 3) + 2 = 6 + (3 + x)$

   c) $(6 + 5) + x = (2 + 5) + 6$

3. Give examples of pairs of operations outside of mathematics which are commutative and also examples which are not commutative.

4. Is it possible for two fathers and two sons to be just three persons? Explain. What is the connection between this and this section?

5. When one adds a column of figures from the bottom up and then from the top down, what properties show that the two results should be the same? Is there a method of finding the following sum which is better than either adding up or adding down:

   $$9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 ?$$

   Could your same method be applied to the following sum:

   $$15 + 14 + 13 + 12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 ?$$

6. Use the above methods to find a short method of adding the multiples of 9 from 9 to 99, inclusive.

7. Let A be the set of people who like ice cream and B the set who like chocolate, all chosen from a given class of students. What would be the interpretation of the formula in Problem 1 below?

8. Give an example of a set S and a relation R, different from those given above, for which S forms an ordered set with respect to the relation R. Then give an example in which S does not form an ordered set.

9. Suppose we define a relationship R between two whole numbers as follows: 1. If a - b is even (divisible by 2), then a R b means $a \geq b$. 2. If a is odd and b even, then a R b is true. Which of the fundamental properties of order does R have?

Problems

1. If A and B are two sets and $n(P)$ denotes the number of elements in any set P, prove

$$n(A) + n(B) = n(A \quad B) + n(A \quad B) ,$$

where A B denotes the intersection of A and B, that is, the set of elements common to both.

2. Using the notation of the previous problem, what is an interpretation in terms of sets of the well-defined property of addition?

3. Use dots to illustrate the associative property of addition, as for the commutative property above.

3.4 Multiplication of whole numbers

If we have five sets each containing three elements and if no two sets have an element in common, then the number of elements in all five sets combined is what we mean by the product of 5 and 3, or 15. In other words, 15 is the sum of five threes:

$$15 = 3 + 3 + 3 + 3 + 3 .$$

In general, for b and c , any natural numbers

$$b \cdot c = c + c + c + \ldots + c$$

where there are b c's in the sum. This is what we mean by the product of two natural numbers. We indicate the product in one of three ways: $b \cdot c$, $b \times c$, bc. Similarly,

$$c \cdot b = b + b + b + \ldots + b ,$$

where the number of b's in the sum is c . It is not at all apparent that the two above sums are equal.

One way to see this for the pair of numbers 3 and 5 is to consider the following array of dots:

. . . . .

. . . . .

. . . . .

If we count the dots by columns, we have five columns of three dots each or five threes. If we count them by rows, we have three rows of five dots each or three fives. The number of dots is independent of the scheme used to count them and so the two results must be equal. It is easy to believe that this same scheme could be used with any pair of natural numbers to arrive at the corresponding result, which we know intuitively from long experience to be true. This is the *commutative property* for multiplication of natural numbers:

$$bc = cb$$

where bc denotes the product of the numbers b and c.

Notice that this property allows us to speak of the product of two numbers without ambiguity and to check a multiplication by performing it in the other order.

We have also shown almost incidentally, the *closure property* for multiplication:

If b *and* c *are two natural numbers, then* bc *is a natural number.*

Notice that multiplication is possible only when we are repeating in a sum the same number again and again.

What of multiplication by zero? We would want five zeros to be zero: $0 + 0 + 0 + 0 + 0 = 0$, and in general $x \cdot 0 = 0$. But zero x's has no meaning. However, since we want multiplication to be commutative for 0 as well as for the natural numbers, we *define* $0 \cdot x$ to be 0. Then we have

$$bc = cb$$

for every pair of whole numbers. Also *the set of whole numbers is closed under multiplication*.

Notice, however, that unless one of b and c is zero, their product cannot be zero. In other words, if for two whole numbers, b and c, $bc = 0$ or $cb = 0$, then $b = 0$ or $c = 0$ or both.

The *well-defined property* for multiplication is quite obvious here, but we state it for future reference:

> If a and b are two names for the same whole number and c is a whole number, then $ac = bc$.

The associative property for whole numbers is harder to justify, and perhaps it is better just to assume it outright. However, there is a three-dimensional model that might make it seem plausible. Consider a stack of four trays of glasses, where in each tray the arrangement is like that in the rectangular array of dots above. If we count the glasses by trays, we have $4 \cdot (5 \cdot 3)$. If we look at the stack from the front, we see four rows of five glasses each, and we know there are two similar arrays behind it, giving us $(4 \cdot 5) \cdot 3$. These results must be the same. This is an example of the *associative property* for multiplication of whole numbers:

$$a(bc) = (ab)c$$

for any whole numbers a, b and c. At the end of the next section we shall see that the associative property can be proved from the distributive property.

Here again we would have to give the number zero special attention, but we need merely notice that if any one of a, b or c is zero, both products are zero. Just as for addition, the associative property allows us to find the product of more than two numbers and to write abc without any ambiguity.

Recall that in the previous section we called the number zero the identity element for addition because $0 + b = b + 0 = b$, no matter what whole number b is. The corresponding number for multiplication is 1 since

$$1 \cdot b = b \cdot 1 = b .$$

This number is called the identity element for multiplication.

3.5 The distributive property

This is the property which seems to be the least understood of all the elementary properties of numbers, and yet it is used unconsciously whenever we multiply two numbers one of which is greater than ten. For instance, to find the product of 2 and 34, we multiply 2 by 4 and get the units digit and 2 by 3 to get the tens digit. What we are really using is: $2(30 + 4) = 2 \cdot 30 + 2 \cdot 4$.

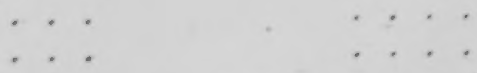
In general the distributive property for whole numbers is

$$a(b + c) = (ab) + (ac) = (b + c)a$$

where the second equality follows from the commutative property for multiplication. It is called by this name since in a sense the multiplication is distributed through the members of the sum. It is also used in the other direction. For instance, to find the sum

$$2 \cdot 27 + 2 \cdot 33 ,$$

it would be easier to add 27 and 33 and multiply the result by 2. This is the factoring process of algebra. This property can be seen by an array of dots as follows:



Reading by rows, we have $2(3 + 4)$ or taking the two parts separately, we would have $2 \cdot 3 + 2 \cdot 4$.

When we multiply two numbers greater than ten, we actually use the distributive property twice, as is illustrated by the following:

$$(21)(34) = (21)(30 + 4) = 21 \cdot 30 + 21 \cdot 4 =$$
$$(20 + 1) \cdot 30 + (20 + 1) \cdot 4 = 20 \cdot 30 + 1 \cdot 30 + 20 \cdot 4 + 1 \cdot 4 = 600 + 30 + 80 + 4 .$$

Notice that we used both orders for the distributive property and also the associative property for addition. In letters this would give us

$$(a + b)(c + d) = ac + ad + bc + bd,$$

where, of course, on the right side it is understood from the convention that we calculate the products before we add.

One of the characteristics of the distributive property is that there are many ways in which it may be misinterpreted. Note, for instance, that $3 \cdot 5 + 2 \neq 3(5 + 2)$. Also the distributive property does not hold for multiplication alone: $(2 \cdot 3)(2 \cdot 5) \neq 2 \cdot 3 \cdot 5$.

The distributive property is implicit in most of our arithmetic calculations. We shall be using it in the tests for divisibility in the following chapter. Moreover, in algebra it is the fundamental property at the basis of factoring. For instance, to find the product

$$(a + b)^2$$

in algebra, we use the distributive property twice as follows:

$$(a + b)(a + b) = (a + b)a + (a + b)b = a^2 + ba + ab + b .$$

(We have also used the associative property of addition.) To complete the result, we use the commutative property for multiplication to get $ab = ba$ and have as our final result

$$a^2 + 2ab + b^2 .$$

Furthermore, to factor this expression we use the distributive property in the reverse direction. So it is especially important that, at this stage when we are working with familiar numbers, this property be made evident and natural.

It is possible to prove the associative property for whole numbers by use of the distributive property. With the thought that the teacher might be interested to see how it goes, we include such a proof at this point. We want to prove:

$$(ab)c = a(bc)$$

for all whole numbers a , b , and c . To accomplish this, we prove it in succession for various values of a . First, if $a = 1$, the equality becomes

$$(1b)c = 1(bc) ,$$

which is true since both sides are equal to bc. If $a = 2$, we have

$$(2b)c = (b + b)c = bc + bc$$

on the one hand, and

$$2(bc) = bc + bc$$

on the other.

Now we want to prove it for $a = 3$, that is $a = 3 = 2 + 1$. Then our desired equality is

$$[(2 + 1)b]c = (2 + 1)(bc) .$$

The left side is, using the distributive property,

$$(2b + b)c = (2b)c + bc .$$

The right side, using the same property, is equal to

$$2(bc) + bc .$$

But we have already shown that $(2b)c = 2(bc)$, and hence

$$[(2 + 1)b]c = (2b)c + bc = 2(bc) + bc = (2 + 1)(bc) .$$

Our next step would be to prove it for $a = 4 = 3 + 1$. This we could do by carrying through the above proof with 2 replaced by 3. Next we could prove it in the same way for $5 = 4 + 1$ and so for all values of a . However, just to show this a little more formally, let us carry through the proof with 2 replaced by n . That is, we assume

$$(nb)c = n(bc)$$

and want to prove

$$[(n + 1)b]c = (n + 1)(bc).$$

Now the left side is equal to

$$(nb + b)c = (nb)c + bc$$

and the right side to

$$n(bc) + bc.$$

But by our assumption $(nb)c = n(bc)$, and we have

$$[(n + 1)b] = (nb)c + bc = n(bc) + bc = (n + 1)bc.$$

Thus on the assumption of the property for $a = n$, we have shown it for $a = n + 1$.

The advantage of this method is that we have now shown that if the equality holds for any whole number n , it holds for the next one. So since

It holds for $a = 2$, it holds for $a = 2 + 1 = 3$; since it holds for $a = 3$, it holds for $a = 4 = 3 + 1$, and so on. Some readers may recognize this type of argument as mathematical induction.

Problems

1. For each of the following find for what whole numbers--a, b, c--the equalities hold:

   a) $(ab) + c = ab + ac$ b) $(ab) \cdot (ac) = abc$

   c) $(ab) + (ca) = a(b + c)$

2. Prove $a(b + c + d) = ab + ac + ad$, pointing out at each stage just what properties you are using.

3. Dissect the process of multiplying 23 by 78 using the distributive property and others. At each step point out what properties you are using. Compare the process for $23 \cdot 78$ with that for $78 \cdot 23$.

4. If we divide the number 327,327 first by 7, then by 11, then by 13, the final quotient will be 327. Is this only by chance, or will it be true that if we divide any number of the form abc,abc by 7, 11 and then 13, our final quotient will be abc ? Why or why not?

3.6 Subtraction

We have many examples of inverse operations inside and outside of mathematics. A man puts on a coat or takes it off. One undoes the other, and sometimes it can be done and sometimes not. (He cannot take off the coat if he does not have it on.) When we add, we solve an equation $a + b = x$; that is, we are given the numbers and want to find the sum. When we subtract, we are given one of the numbers and the sum and want to find the other number; that is, we solve the equation $a + x = b$. This is illustrated by the usual process of making change when the clerk starts with the amount of the purchase and gives you money until it reaches the amount which you gave him. When we solve $a + x = b$, we write the solution as: $x = b - a$. And since $x + a = a + x$, the answer is the same when we solve $x + a = b$. In words, x is the number which, when added to a , gives b .

Now if $a = 7$ and $b = 5$, there is no whole number which we can add to 7 to get 5, and the equation $7 + x = 5$ has no solution in whole numbers. In fact, we have the following possibilities for the solution of $a + x = b$:

1. If $a > b$, no solution. 2. If $a = b$, $x = 0$.

3. If $a < b$, $x = b - a$.

Later of course we shall extend our number system so that we shall have a solution in the first case also, but at this stage if we are to consider only whole numbers, our only choice is to say that in our number system we cannot subtract a from b if a is greater than b.

What are some of the properties of subtraction? If $b > a$, we have $a + (b - a) = (b - a) + a$ because both a and $(b - a)$ are whole numbers. But what about the commutative property for subtraction:

$$b - a = a - b?$$

This can hold only if $a = b$ when both sides are zero, for if $b > a$, the left side is a whole number but the right side is not, while if $a > b$, the right side is a whole number but the left side is not.

What of the associative property? Compare $9 - (7 - 2)$ with $(9 - 7) - 2$. The former is equal to 4 and the latter to zero. This means that without some understanding the expression $9 - 7 - 2$ is ambiguous. It is customary to adopt the second meaning, but this of course makes it very important that the uninitiated are instructed in this mathematical mystery. Is it any wonder that some students think that to compute $2 + 7 \cdot 9$ you add 2 to 7 and multiply the sum by 9! For this reason there are those who would like to discontinue the convention that $a - b - c$ means $(a - b) - c$.

What of the distributive property? Let us try this out: $9(7 - 2) = 9 \cdot 7 - 9 \cdot 2 = 45$. Let us try this in general to find

$$a(b - c)$$

where $b > c$. We know that there is a whole number x so that $b = c + x$. Then $ab = a(c + x) = ac + ax$ by the distributive property for whole numbers. But this shows that ax is the number which you add to ac to get ab. In other words, $ax = ab - ac$. This is then equal to $a(b - c)$, since $x = b - c$. So the distributive property does hold for subtraction, at least for a to the left of the parenthesis. See Problem 1 below.

Exercises

1. Introduce parentheses in the following to make the equation true:
   a) $8 - 5 - 3 = 6$ b) $8 - 5 - 3 = 0$
2. How many possible different numbers can one represent by placing one pair of parentheses in the following?

   $$32 - 16 - 8 - 4 - 2 - 1$$
3. Introduce parentheses in the following to make the equation true:
   a) $9 \cdot 7 + 3 = 90$ b) $9 \cdot 7 + 3 = 66$

4. How many possible different numbers can one represent by placing one pair of parentheses in each of the following:

   a) $8 \cdot 2 + 7 \cdot 3$ b) $9 - 2 \cdot 3 - 1$

5. What properties of numbers are used to establish the following for any two numbers x and y :

$$(x - y)(x + y) = x^2 - y^2, \text{ assuming } x > y.$$

   By this or other means find an expression equal to

$$(x + 1)^2 - x^2 .$$

6. Write the sequence of squares and compute the difference between each square and the next, as follows:

   1 4 9 16 25 36 49 64 81 100 ...
   3 5 7 9 11 13 15 17 19 ...

   Use the results of Exercise 5 to find a relationship between the differences and the adjacent squares.

7. The sum of the first four odd numbers, 1 + 3 + 5 + 7, is 16, the square of 4. The sum of the first nine odd numbers is 9^2. Why would a similar result hold in general?

8. Suppose instead of considering all the squares, we considered only the odd squares: 1, 9, 25, 49. What can you find about the differences similar to that in Exercise 6?

9. Perform on a given pair of numbers two sets of operations as follows for the numbers 11 and 7.

I	II
$11 - 7 = 4$	$11^2 = 121$
$11 + 7 = 18$	$7^2 = 49$

   Then the product of the two results in I, $4 \cdot 18 = 72$, is the same as the difference of the two results in II, $121 - 49 = 72$. Is this an accident, or will it hold for any two numbers? Why?

10. If we divide 45,624,562 by 73 and then by 137, we get 4,562. Is this an accident?

11. A girl went to the pantry with only a 5-cup and a 3-cup container to get 4 cups of flour. Can this be done if nothing but the flour container is used in addition to the two containers? If so, how? Explain the use of parentheses in your answer.

Problems

1. If a , b , c are whole numbers and b > c, is the following true?

$$(b - c)a = ba - ca$$

2. If a , b , c are whole numbers and b > c, is the following true?

$$a + (b - c) = (a + b) - c$$

If not, give an example. If so, prove it.

3. Answer the question in the previous problem for the expressions

$$a - (b + c) = (a - b) + c$$

3.7 Cancellation properties

At this point the cancellation properties can be thought of as the converse of the well-defined properties, though in the light of later number systems we shall see that all can be thought of as well-defined properties. In Section 3.3 the following important property of addition was pointed out for whole numbers a , b , c :

$$\text{If } a = b, \text{ then } a + c = b + c .$$

The cancellation property for addition is the converse:

$$\text{If } a + c = b + c, \text{ then } a = b .$$

This follows immediately because if we let $a + c = x$, then by definition $a = x - c$, and since x is also equal to $b + c$, we have $b = x - c$. Hence $a = b$. In effect what we have done is to subtract c from both sides of the equation $a + c = b + c$.

For multiplication, the well-defined property was: for whole numbers a , b , c

$$a = b \text{ implies } ac = bc .$$

What of the converse statement:

$$ac = bc \text{ implies } a = b?$$

This is certainly not true if $c = 0$, since $3 \cdot 0 = 2 \cdot 0$, but $3 \neq 2$. But if $c \neq 0$, it is true. Why? The following sequence of steps leads to the desired result:

$$ac = bc \text{ means } ac - bc = 0 .$$

From the distributive property for subtraction we have:

$$(a - b)c = 0.$$

But we know that the product of two whole numbers can be zero only if one of them is zero. Hence the last equation implies

$$a - b = 0 \quad \text{or} \quad c = 0 .$$

Thus if $c \neq 0$, $a - b = 0$; that is, $a = b$. Thus we have proved the *cancellation property for multiplication*:

$$ac = bc \text{ with } c \neq 0 \text{ implies } a = b .$$

Problem

1. Prove that if a , b , c are whole numbers with $c \neq 0$, then $ca = cb$ implies $a = b$.

3.8 Division

Recall that subtraction is the inverse operation of addition. That is, when we add we solve the equation $a + b = x$, and when we subtract, we solve the equation $a + y = b$. In other words, y is the number we add to a to get b. Sometimes there is such a number in the set of whole numbers and sometimes not. When there is such a number, we call it $b - a$, and we say we get it by subtracting a from b.

Similarly, for multiplication we solve the equation $ab = x$, and for division we solve the equation $ay = b$. In other words, y is the number we multiply by a to get b . Sometimes there is such a number in the set of whole numbers and sometimes not. When there is such a number, we call it

$$b/a \quad \text{or} \quad \frac{b}{a}$$

and say that we get it by dividing b by a . Just as subtraction and addition are inverse operations, so are multiplication and division.

Another way to think of the inverse operation is in terms of one "undoing" the other. For instance, if $b - a$ exists, then

$$b - a + a = b$$

because $b - a$ is the number which, when added to a , gives b . Similarly, if $\frac{b}{a}$ exists,

$$\frac{b}{a} a = b$$

because by definition, $\frac{b}{a}$ is the number which, when multiplied by a , gives b.

To repeat, sometimes there is an inverse number for multiplication and sometimes not. For instance, $3x = 12$ has a solution, $x = 4$, but $5x = 12$ has no solution in the set of whole numbers. Here it is not so easy to tell by inspection whether or not a solution exists, but we have a term which we can use in this connection. If $ax = b$ has a solution in whole numbers, we say that b is *divisible* by a, or that b is a *multiple* of a, or that a is a *divisor* of b or a is a *factor* of b.

Since in the next chapter we shall be concerned with properties connected with divisibility or non-divisibility, we shall not carry this further at this point, except to write that if b is divisible by a, we write the solution of $ax = b$ in one of several ways:

$$b/a\ ,\quad b \div a\ ,\quad \frac{b}{a}\ .$$

In the process of division the identities for addition and multiplication play a special role. If $a = 1$, the equation $ax = b$ always has a solution, $x = b$. But if $a = 0$, then $ax = 0$ for every whole number x, and one of two things can happen: 1. If $b \neq 0$, there is no number x for which $0 \cdot x = b$. 2. If $b = 0$, any x will do. Thus we either have no solution or too many. So we must be careful to avoid division by zero. We shall see that not only for whole numbers but for any numbers there would be this trouble with division by zero.

One can also look at this question of divisibility somewhat physically. The number bc came from having b sets with c elements in each set. Then to solve $bx = c$, we want to find b sets with x elements in each set such that the total number is c. To solve $xb = c$, we would want to find x sets with b in each set so that the total number is c. If one equation is solvable, so is the other from the commutative property of multiplication (though the physical situation is quite different), and the two solutions are the same number.

The symbols $a \div b \div c$ and $a/b/c$ are ambiguous, and parentheses must be used to give them meaning. An example is sufficient to show why they are ambiguous: $(8/4)/2 = 2/2 = 1$, while if we put the parentheses in differently, we have $8/(4/2) = 8/2 = 4$.

Exercises

1. Consider the equality:

$$(a * b) * c = a * (b * c)$$

where a, b and c are whole numbers. Why does this hold for all whole numbers when * stands for + as well as when * stands for multiplication? Suppose * stands for - and a - b, as well as (a - b) - c, is a whole number; is the right side necessarily a whole number, and if so, is it equal to the left? Answer the corresponding questions for division instead of subtraction.

2. Answer the same questions as for Exercise 1 for the expression

$$a * (b * c) = c * (b * a) .$$

3. How many possible different numbers can one represent by placing parentheses in

$$16 \div 8 \div 4 \div 2 \div 1 \ ?$$

1. In each of the following a, b and c are whole numbers and every indicated quotient is an integer. For instance, in part 1) on the left side, b is divisible by a, and c is divisible by their quotient; no denominators are zero. Indicate which of the equalities hold:

a) $c/(b/a) = (c/b)/a$
b) $c/(b + a) = c/b + c/a$
c) $(c + b)/a = c/a + b/a$
d) $(c - b)/a = c/a - b/a$ with $c > b$

3.9 Inequalities

There are also properties of inequalities which correspond to the well-defined and cancellation properties for equality. In this slightly less familiar setting, it is perhaps more apparent why we should take notice of these properties. If Johnny has fewer marbles than Henry and if we give each of them the same number of additional marbles, Johnny will still have fewer marbles than Henry. Also if we take the same number of marbles away from both, Johnny will still have fewer. In notation this means:

i) If $a < b$, then $a + c < b + c$
ii) If $a + c < b + c$, then $a < b$.

We can just accept these as fundamental properties of numbers. But if we define inequality in terms of addition, we can derive these properties from previous ones. This we shall do.

Definition: If for some natural number x it is true that $a + x = b$, we say that "a is less than b" and write it $a < b$, or alternately, "b is greater than a," which is written $b > a$.

Note that a definition works both ways; that is, in this case by the statement $a < b$ we mean that there is a natural number x such that $a + x = b$, as well as in the other order given in the definition. We shall extend this definition later in the book to other numbers. Now, in terms of this definition, let us return to the first of the statements above and show how, using the definition, we can prove:

$$\text{If } a < b \text{, then } a + c < b + c.$$

Proof: From the hypothesis, $a < b$, we know that there is a natural number x such that $a + x = b$. But by the well-defined property for addition

$$(a + x) + c = b + c .$$

By the commutative property and associative property for addition

$$(a + c) + x = b + c ,$$

which by the definition is equivalent to

$$a + c < b + c .$$

This proves statement i). We leave the proof of ii) for a problem below.

We can use the same technique to deal with inequalities for multiplication. The corresponding ones are for natural numbers a, b, c

iii) If $a < b$, then $ac < bc$

iv) If $ac < bc$, then $a < b$.

To prove iii) notice that $a < b$ is equivalent to $a + x = b$. Then by the well-defined property for multiplication,

$$(a + x)c = bc .$$

By the distributive property

$$ac + xc = bc .$$

Since the set of natural numbers is closed under multiplication, then xc is a natural number, and the last equation implies $ac < bc$.

We leave the proof of iv) to the problems. The reader should be warned that while i) and ii) hold for larger sets of numbers, iii) and iv) do not hold in general. In fact, if we were considering whole numbers instead of natural numbers, we would have had to specify $c \neq 0$ in iii) and iv).

The teacher might well ask at this point, why give these proofs when we have assumed less obvious results. The author's reason was to emphasize the connection between inequality and equality, noting that the former could be defined in terms of the latter and that the well-defined properties of the latter imply those of the former. It is not so much the proofs in themselves but the relationships between inequality and equality which are important.

Exercises

1. If a, b, c and d are natural numbers, show that $a < b$ and $c < d$ implies $ac < bd$.
   (This can be done either from the definition of inequality in terms of sums or from the basic properties of inequality; note the summary in Section 3.10.)

2. If a, b, c and d are whole numbers, show that $a < b$ and $c < d$ implies $a + c < b + d$.

3. Suppose $a < b$ and $a + c < b + d$. What conclusions could be drawn, if any, about the relative size of c and d? (a, b, c, d are whole numbers.)

4. Suppose $a < b$ and $a + c > b + d$. What conclusions could be drawn, if any, about the relative size of c and d? (a, b, c, d are whole numbers.)

5. Answer the same questions as in Exercises 3 and 4 when the sums are replaced by products, if we assume that a, b, c, d are natural numbers.

6. State inequalities on a, b and c so that each of the following are equal to whole numbers, assuming that a, b and c are themselves whole numbers:

$$a - (b - c)\ ,\quad (a - b) - c.$$

What are the conditions that both represent natural numbers?

Problems

1. Prove, using the methods above, that $a + c < b + c$ implies $a < b$.

2. Prove that for natural numbers a, b, c ($c \neq 0$),

$$ac < bc \text{ implies } a < b\ . \quad \text{(The cancellation property)}$$

3.10 Looking backward

The reader at times in this chapter may have wondered why the seeming effort to make something obvious difficult. The point is that these properties are not so obvious in later situations. It is better to dig into them here so that later we will have seen them before. The author does not recommend that all these properties be pointed out to the students. But the teacher should have in mind that these are the fundamental properties and that the students should have intuitive experiences which will make these properties appear. The matter of first importance is that they shall know these properties. Much of what lies back of them and why they are related will of necessity come later. Time spent here should pay dividends later on. But again, these ideas will recur again and again, and as this happens, familiarity with them should increase.

Finally, for ease of reference, we list here the properties of whole numbers which we have dealt with in this chapter. Here the letters stand for whole numbers.

Fundamental Algebraic Properties

1. Closure properties: $a + b$ and ab are whole numbers.
2. Commutative properties: $a + b = b + a$ and $ab = ba$.
3. Associative properties: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
4. Existence of an identity element: For addition it is 0 which has the property: $0 + a = a + 0 = a$ for all a. For multiplication it is 1 and has the property: $1 \cdot a = a \cdot 1 = a$ for all a.
5. The distributive properties: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.
6. The well-defined properties: $a = b$ implies $a + c = b + c$ and $ac = bc$.
7. The cancellation properties: If $a + c = b + c$, then $a = b$.

If $ac = bc$ and $c \neq 0$, then $a = b$.

An important consequence of the above is that a product of two whole numbers is zero if and only if one or both of them is zero.

Properties of Inequality

11. If b and c are any two whole numbers, then exactly one of the following holds: $b > c$, $b = c$, $c > b$.

2I. If $a > b$ and $b > c$, then $a > c$. (The transitive property)

3I. The well-defined properties: If $a > b$, then $a + c > b + c$; and, if $c \neq 0$, then $ac > bc$.

4I. The cancellation properties: If $a + c > b + c$, then $a > b$; and, if $c \neq 0$ and $ac > bc$, then $a > b$.

As we noted, the conclusions for multiplication in 3I and 4I are not valid for other sets of numbers. As it turns out, all the other properties hold for integers, rational numbers, and real numbers as well. In fact, the algebraic properties hold for complex numbers and other mathematical systems, as we shall see.

References

1 (Chapters 5, 8, 9), 8 (Chapter 3), 22 (Chapters 4, 6, 7).

Chapter 4

DIVISIBILITY AND PROPERTIES OF WHOLE NUMBERS

4.1 Introduction

The properties of whole numbers which we considered in the previous chapter are often called "algebraic" because they are properties which also hold for other kinds of numbers and the elementary processes of algebra. But there are also properties, often called the arithmetic properties, which arise from the restricted character of the whole numbers. Subtraction for whole numbers is made possible by the introduction of negative numbers to form the set of integers. But division is quite a different matter for two reasons. First, the criteria for divisibility are more complex than for "subtractibility." Second, when we make division possible by the invention of the rational numbers, we lose some of the essential properties of the integers, for example, the lack of the property of density. Just as outside of mathematics the lack of something important makes it the center of attention (like water in desert country), so the fact that divisibility is a problem for the whole numbers forces upon us a consideration of this property.

Consideration of these arithmetic properties is important not only because these are fundamental to our understanding of the integers (whole numbers and negative numbers) but also because many of the same properties carry over to the polynomials which are studied later on in algebra. Also it is true that much of our life is concerned with whole numbers: populations, itemization, coinage (whole number multiples of one cent), for instance. In fact, in many ways the physical world is discrete rather than continuous. Finally, many of the puzzles which amateur mathematicians fancy have whole numbers for answers. Moreover, there is a branch of mathematics called the Theory of Numbers which concerns itself almost exclusively with the properties of integers.

Except for the last section of this chapter, we shall be concerned only with the whole numbers. We shall see in the last section that there is no great difficulty in carrying over our results to the complete set of integers including the negative integers.

4.2 *Divisibility*

When we say that 21 is divisible by 3, we mean that there is an integer, 7, such that 21 is the product of 3 and 7. In general we have the

Definition: r is divisible by s if there is an integer x such that xs = r, and we write $x = \frac{r}{s}$ or $x = r/s$. Physically this can have two meanings. From one point of view it means that we can apportion r objects into s sets with the same number, x, in each set. From another point of view, as was pointed out in Section 3.8, since multiplication is commutative, it means that the r objects can be apportioned into a certain number of sets, x, with s objects in each set. Actually this is a physical way of determining divisibility. For instance, to determine whether a number n is divisible by 3, we form from n objects one set of 3, another set of 3, another set of 3, until one of two things happens: either we have none left over, in which case n is divisible by 3, or we have something left over less than 3, in which case n is not divisible by 3. We know that for any whole number n, not a multiple of 3, there will be two successive multiples of 3 between which it lies. This result which we call the *remainder property* is worth stating formally:

If b *and* c *are any natural numbers, there are whole numbers* q *and* r *such that* $b = cq + r$, $r < c$. In fact, q is the quotient when b is divided by c and r is the remainder. From another point of view, q is the greatest integer less than or equal to $\frac{b}{c}$ because $\frac{b}{c} = q + \frac{r}{c}$ and $0 \leq \frac{r}{c} < 1$. Thus q, and therefore r, is unique. To say that *b is divisible by c* is the same as saying that the remainder is zero.

There are many different ways to express the facts of divisibility. Each of the following list is a different way of saying that r is divisible by s.

a) s is a factor of r.

b) s is a divisor of r.

c) r is a multiple of s.

d) r is divisible by s.

In this connection the number one occupies a unique position. It is a divisor of every whole number, and its only divisor is itself (recall that we are restricting ourselves to whole numbers). On the other hand, zero is not a divisor of any number but is divisible by all.

There are some fundamental properties of divisibility which we now list, leaving the proofs as exercises.

1. If r is a factor of s and s is a factor of r, then s = r.

2. If r is a factor of s and s is a factor of t, then r is a factor of t. (The transitive property--see Section 3.2)

3. If r is a factor of s and of t, it is a factor of $s + t$ and st; and, if $s > t$, of $s - t$.

Of course, the letters above stand for counting or natural numbers. A whole number divisible by 2 is called an even number, and one not divisible by 2 an odd number. Property 3 above shows us that the sum of two even numbers is an even number.

Exercises

1. Find the number of factors of:
   a) 9, 49, and $9 \cdot 49$
   b) 5, 3, 4, and 60
   c) 15, 21, and $15 \cdot 21$

2. Can you make any guesses from the examples of the previous exercise? If so, try to establish them by proof.

3. Prove that if b is a factor of c and d is a factor of f, then bd is a factor of cf.

4. Find q and r for each of the following pairs of values of b and c, assuming that in each case $b = cq + r$, with c and r whole numbers and $r < c$.
   a) $b = 17$, $c = 5$ b) $b = 379$, $c = 23$

5. Let b and c be any two numbers, and show that there is a multiple of c which is not farther from b than $\frac{c}{2}$. If the difference between b and the nearest multiple of c is $\frac{c}{2}$, show that $2b$ is a multiple of c. (The answers to these questions might be easier if you try them out for particular numbers first.)

6. How many different remainders are possible when dividing a number by 73? If the sum of two numbers is divisible by 73, what can you say about the remainders when these two numbers are divided by 73?

7. If a and b are two numbers and if $a + b$ as well as $a - b$ is divisible by 73, show that a and b are divisible by 73. Would the same conclusion follow if 73 were replaced by any other number? Why or why not?

8. Show that if x and y are whole numbers and n is a whole number such that

$$n = x^2 - y^2,$$

then $(x - y)$ and $(x + y)$ are factors of n. Are there solutions for $n = 15$ and $n = 22$? Where there are solutions, find them.

9. Show that the remainder when any number is divided by 3 is one of 0, 1, or 2. Hence show that if x is an integer not divisible by 3, then $x - 1$ or $x + 1$ is divisible by 3 and hence that $x^2 - 1$ is divisible by 3. Show that if x^2 and y^2 are the squares of two natural numbers, either one is divisible by 3 or their difference is divisible by 3.

Problems

1. In terms of the number line, give a geometrical interpretation of $b = cq + r$, where r is a whole number less than c.

2. Prove the three properties of divisibility.

3. Prove that if when whole numbers x and y, $x > y$, are divided by s the remainders are the same, then $x - y$ is divisible by s.

4. Prove that if x and y are whole numbers and if $x - y$ is a whole number divisible by s, then the remainders when x and y are divided by s are the same.

4.3 Prime numbers

We have seen that 1 is a factor of every counting number. And every counting number has itself as a factor as well. Numbers which have only these two factors are called prime numbers. In other words, a prime number is a counting number which has just two factors: itself and 1. The number 1 is not counted among the prime numbers for an important reason which we shall explain below. Counting numbers different from 1 which are not prime numbers are called composite numbers. Thus the set of counting numbers consists of three subsets:

a) The number 1 which has just one factor.
b) The prime numbers which have just two factors.
c) The composite numbers which have more than two factors.

The first ten prime numbers are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

Notice that only the first one is an even number. Why? Also the only two successive numbers which are prime numbers are 2 and 3. Why?

To many mathematicians prime numbers are important because of their irregularity and all that we do not know about them. They are important for children to know about because in a sense they are the building blocks for the structure of the integers. This stems from a fundamental result which we state as a theorem:

Theorem 1. (The Unique Factorization Property of Counting Numbers, or, The Fundamental Theorem of Arithmetic) First, every counting number can be expressed as a product of prime numbers, and second, the product is unique except for the order of the factors.

The first part of this result is easy to show; the second depends on a process called the euclidean algorithm which we discuss in Section 4 of this chapter. The proof of this theorem is completed in Section 8 of this chapter.

To show that such a factorization is possible, consider a counting number n. If it is not a prime number, it has factors between 1 and itself, and it can be written in the form $n = rs$ where neither r nor s is 1. If both r and s are primes, we have the result needed. If r, for instance, is not a prime number, it can be written as a product of two other numbers. So we can continue this process. It will have to stop because every time our numbers are smaller than before. Hence in the end we will have expressed n as a product of prime factors.

Consider the number 2275. It is divisible by 5 since the last digit is. (See Section 4.6.) Dividing by 5, we have $2275 = 5 \cdot 455$. Then again $455 = 5 \cdot 91$, and $2275 = 5 \cdot 5 \cdot 91$. For 91 we see that 3 is not a divisor and neither is 5, but it is divisible by 7, the next prime, and we now have

$$2275 = 5 \cdot 5 \cdot 7 \cdot 13 .$$

So 2275 is a product of four prime numbers, two of which are the same. Of course we could have done this another way. We could have divided by 7 first, having $2275 = 7 \cdot 325$. Then divisions by 5 and 13 give

$$2275 = 7 \cdot 5 \cdot 13 \cdot 5 .$$

The factorization is different but only in the order in which the factors appear; in both there are two 5's, one 7, and one 13.

Let us choose an example somewhat harder than that above, the factorization of the number 551. We would try in succession 3, 5, 7, 11, 13, 17 without success, but 19 is a factor and $551 = 19 \cdot 29$. In general, would one have to try all the primes less than the given number before finding out whether it is prime or not?

Now it is clear why we wanted to exclude 1 from the list of primes. If it were included, we would have many trivially different ways of expressing a number as a product of prime numbers, using as many one's as we wish. There are other reasons for excluding 1, but this is the most immediate.

A somewhat more compact way of writing a number as a product of its prime factors is to use exponents as in Chapter II. For instance,

$$2275 = 5^2 \cdot 7 \cdot 13 .$$

In letters we might have

$$n = p^a q^b r^c$$

where p, q, and r are distinct prime numbers and a, b, c are counting numbers.

Exercises

1. Why is 2 the only even prime number?
2. Let x and y be two successive counting numbers; that is, their difference is 1. Can both of them be prime numbers? If so, under what conditions?
3. Let n, $n + 1$ and $n + 2$ be three successive counting numbers. Can they all be prime numbers? Give reasons for your answer.
4. Let n and $n + 2$ be two counting numbers. Can they both be prime numbers? Give reasons.
5. Let n, $n + 2$ and $n + 4$ be three counting numbers. Can they all be prime numbers? Give reasons.
6. Applying Problem 1, find which of the following are prime numbers:

   313, 323, 4501

Problem

1. How far would one have to try possible prime factors of 4501 before ascertaining whether or not it is a prime? What is the general result?

4.4 Greatest common factor

So far we have been considering factors of one number at a time. For two numbers we may find that they have factors in common, that is, common factors. Every pair will have 1 as a common factor, and sometimes there are more. Let us list all the factors of three numbers:

12: 1, 2, 3, 4, 6, 12
56: 1, 2, 4, 7, 8, 14, 28, 56
175: 1, 5, 7, 25, 35, 175

The pair 12, 56 has common factors: 1, 2, 4.
The pair 12, 175 has no common factor except 1.
The pair 56, 175 has just two common factors: 1 and 7.

In each case we can pick out the greatest common factor (abbreviated to g.c.f.). (It is sometimes called the greatest common divisor and abbreviated to g.c.d.) In the first case it is 4, in the second 1, and in the third 7.

Let us look at this discussion in terms of sets. Let F_{12} be the set of factors of 12, F_{56} the set of factors of 56, and F_{175} those of 175. Then

$$F_{12} = \{1, 2, 3, 4, 6, 12\} ,$$
$$F_{56} = \{1, 2, 4, 7, 8, 14, 28, 56\} ,$$
$$F_{175} = \{1, 5, 7, 25, 35, 175\} .$$

Then

$$F_{12} \cap F_{56} = \{1, 2, 4\}$$

is the set of factors common to the first two sets. Similarly

$$F_{12} \cap F_{175} = \{1\} \text{ and } F_{56} \cap F_{175} = \{1, 7\} .$$

Then the greatest common factor for each pair of numbers is the greatest integer in each of the intersection sets. In other words, 4 is the greatest integer in $F_{12} \cap F_{56}$ and hence is the g.c.f. of 12 and 56. Similarly 7 is the g.c.f. of 56 and 175, while the greatest integer in $F_{12} \cap F_{175}$ is the only number it contains, namely 1.

While the above process is very useful in fixing the idea of what the greatest common factor is, it is not the most efficient way of finding it. For small numbers the best way is probably to express each number as a product of its prime factors and from this determine the g.c.f. (greatest common factor). For example, to find the g.c.f. of 525 and 4455, first express each as a product of primes:

$$525 = 3 \cdot 5 \cdot 5 \cdot 7 = 3 \cdot 5^2 \cdot 7; \quad 4455 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 11 = 3^4 \cdot 5 \cdot 11.$$

The highest power of 3 occurring in both is the first power, and similarly for 5. The product of these is 15, the greatest common factor.

In more general terms, suppose

$$m = p^a q^b r^c u^d , \qquad n = p^e q^f t^g$$

where p , q , r , u , t are different prime numbers. The greatest common factor would be $p^h q^j$ where h is the smaller of a and e , and j is the smaller of b and f .

For larger numbers there is a process called the euclidean algorithm based on the remainder property of Section 4.2 which we illustrate by a numerical example. To find the g.c.f. of 299 and 221, divide the larger by the smaller and compute the remainder. Then divide 221 by the remainder and find its remainder. Continue this process until a remainder 0 appears. The previous remainder is the g.c.f. Here is the calculation:

$$\begin{aligned} 299 &= 1 \cdot 221 + 78 \\ 221 &= 2 \cdot 78 + 65 \\ 78 &= 1 \cdot 65 + 13 \\ 65 &= 5 \cdot 13 + 0 . \end{aligned}$$

Conclusion: 13 is the g.c.f. of 299 and 221.

Why does this process give the g.c.f.? We answer this question in terms of the example. The first equation shows that any common factor of 299 and 221 is a factor of 78, since $78 = 299 - 1 \cdot 221$. The second equation shows similarly that any common factor of 221 and 78 is a factor of 65; that is, any common factor of 299 and 221 is a factor of 65. The third equation shows that any common factor of the two given numbers is a factor of 13. On the other hand, starting with the last equation, 13 is a factor of 65; from the third equation it is a factor of 78; from the second equation a factor of 221; and from the first a factor of 299. Thus every common factor of 299 and 221 is a factor of 13, and 13 [illegible] a factor of 299 and 221. Hence 13 is the greatest common factor. From this example it may be seen that this process always gives the g.c.f. of two numbers.

Based on the above ideas, it is possible to devise a shortened means of computation. But for us the computation of a g.c.f. is not very important. It is the existence of a g.c.f. which is important because from this we can prove the Fundamental Theorem of Arithmetic. We should elaborate on this point. The existence of the g.c.f. follows directly from the Fundamental Theorem of Arithmetic, as we have illustrated above, and this is probably the best way to show the existence to a class of junior high school students. But also the euclidean algorithm shows the existence of a g.c.f., and from this algorithm may be proved the Fundamental Theorem of Arithmetic. The euclidean algorithm is the more fundamental result. We shall find useful later the following two important consequences of the idea of g.c.f.

Theorem 2: If b and c have g as their greatest common factor, then 1 is the g.c.f. of $\frac{b}{g}$ and $\frac{c}{g}$.

To show this, suppose e is a common factor of $\frac{b}{g}$ and $\frac{c}{g}$. This means $\frac{b}{g} = eb'$ and $\frac{c}{g} = ec'$ for natural numbers b' and c'. Thus $b = geb'$ and $c = gec'$. This shows that ge is a common divisor of b and c. But g we took to be the greatest common factor of b and c. Hence $ge = g$ and $e = 1$. This is what we wanted to show.

Theorem 3: If a is a factor of bd and if a and b have g.c.f. 1, then a is a factor of d.

This is practically obvious from the Fundamental Theorem of Arithmetic, for 1 is the only factor of a which can divide b, and hence all the other factors of a, including a itself, must divide d. With the usual junior high school class the teacher will want to leave it at that. But it is possible to give a proof without the use of the Fundamental Theorem, that is, depending only on the euclidean algorithm. The point of doing this, as we have noted above, is that then we can safely use this theorem to prove the Fundamental Theorem of Arithmetic. In fact, Theorem 3 follows directly from the following result.

Theorem 4: If 1 is the g.c.f. of two whole numbers a and b, then there are integers x and y such that

$$ax + by = 1.$$

Notice that x and y need not be positive; in fact, one of them must be negative if both a and b are greater than 1.

Before proving Theorem 4, we show how it can be used to prove Theorem 3. Then since for Theorem 3, 1 is the g.c.f. of a and b, Theorem 4 tells us that

$$ax + by = 1$$

for integers x and y. Multiply this by d to get

$$adx + bdy = d.$$

But a divides ad and bd and hence by Property 3 of Section 2 of this chapter, a divides the number on the left side; hence it divides d and our proof is complete.

Now we prove Theorem 4. To do this, consider the set S of all the counting numbers which can be written in the form:

$$ax + by .$$

For instance, using $x = y = 1$, one of these numbers is $a + b$. Using $x = 1$, $y = 0$, we have $ax + by = a$. If $b < a$, then we can take $x = 1$ and $y = -1$ and see that the set S contains $ax + by = a - b$. There are many numbers in S. We let m be the least number in S, that is, the least counting number such that

$$ax + by = m$$

for integers x and y. If we can prove that then $m = 1$, our proof of Theorem 4 will be complete. If we can prove that m divides both a and b, then 1 the g.c.f. of a and b would imply $m = 1$. So first we suppose m does not divide b.

We then have

$$b = qm + r, \text{ where } 0 < r < m$$

and q and r are integers. Then we start with the equation

$$ax + by = m,$$

and multiply both sides by q to get

$$qax + qby = mq = b - r ,$$

$$qax + qby - b = -r ,$$

$$-qax - qby + b = r .$$

This can be written:

$$a(-qx) + b(1 - qy) = r.$$

So we have shown that

$$aX + bY = r$$

where $X = -qx$ and $Y = 1 - qy$. Thus X and Y are integers, and r is less than m. But m is the least number of S. This is the contradiction which we need to complete the proof that m must divide b. Similarly we can prove that m must divide a and hence that, since the g.c.f. of a and b is 1, m must be 1.

It may be helpful if we illustrate this for a numerical case. Take $a = 15$ and $b = 4$. Then in the expression $15x + 4y$ we can take $x = 1$ and $y = -3$ to get

$$15 \cdot 1 - 4 \cdot 3 = 15 - 12 = 3 .$$

Thus 3 is an element of the set S of counting numbers expressible in the form $15x + 4y$ where x and y are integers. But it is not the minimum number of S, for we may write $4 = 3 + 1$ or $3 = 4 - 1$, and replacing 3 in the equation by $4 - 1$, we have

$$15 \cdot 1 - 4 \cdot 3 = 4 - 1$$

or

$$15 - 4 \cdot 3 - 4 = -1$$
$$15 - 4 \cdot 4 = -1$$
$$15(-1) + 4 \cdot 4 = 1.$$

So a solution of $15x + 4y = 1$ is $x = -1$ and $y = 4$.

In Section 4.8 we use Theorem 3 to prove the Fundamental Theorem of Arithmetic.

There is an idea in connection with sets which is quite close to the g.c.f. of two numbers. The intersection of two sets can be thought of as the greatest common subset. What is then the subset analogous to 1 for the g.c.f.?

Exercises

1. Express the following number as a product of prime factors: 17,325 .
2. Use the euclidean algorithm to find the g.c.f. of the numbers 17,325 and 407.
3. Using Theorem 4 and the result of the previous exercise, find for what values of c the following equation has integer solutions x and y:

   $$17{,}325x - 407y = c .$$

   What is the least such value of c which is a natural number? Use the methods of the section to find a solution in this case.
4. Prove that if 1 is the g.c.f. of the natural numbers b and c , then every factor of bc can be expressed in exactly one way as a product of a factor of b and a factor of c . For instance, if $b = 9$ and $c = 49$, $21 = 3 \cdot 7$ where 3 is a factor of 9 and 7 of 49. Also $63 = 9 \cdot 7$.
5. Suppose $g > 1$ is the g.c.f. of the natural numbers b and c . Are there some factors of bc which can be expressed in exactly one way as a product of a factor of b and one of c ? Are there some which have more than one such representation?
6. If g is the g.c.f. of a and b, and h the g.c.f. of g and c , show that h is the g.c.f. of a, b, and c.

7. Let the symbol (a,b) stand for the g.c.f. of a and b . Is this symbol associative; that is, is

$$((a,b),c) = (a,(b,c)) \ ?$$

Problems

1. Answer the last question in this section.

2. A shortened process for the g.c.f. is illustrated by the following:

$$299 = 1 \cdot 221 + 78$$
$$221 = 3 \cdot 78 - 13$$
$$78 = 1 \cdot 13 + 0$$

The difference is that in the second line we used the nearest multiple of 78 to 221 instead of the greatest multiple less than 221. Why does this process yield the g.c.f. as well?

3. Use the euclidean algorithm to find the g.c.f. of 89 and 55. What are some of the characteristics of this particular example?

4. Prove that if $a = bq + r$, then the g.c.f. of a and b is equal to the g.c.f. of b and r . Use this to give another proof that the euclidean algorithm yields the g.c.f.

5. Prove that if g is the g.c.f. of a and b , there are integers x and y such that $ax + by = g$.

4.5 Least common multiple

Though each number has a finite number of factors, it has an infinite number of multiples. The least common multiple of two whole numbers is the smallest multiple of both which is different from zero. For instance, the least common multiple of 12 and 56 is 168. That 168 is the l.c.m. (the abbreviation for "least common multiple") can be seen from the factorizations of 12 and 56:

$$12 = 2^2 \cdot 3 \quad \text{and} \quad 56 = 2^3 \cdot 7 \ .$$

The l.c.m. must be a multiple of 2^2 and 2^3, but since the former is a factor of the latter, we can merely specify that the l.c.m. must be a multiple of 2^3. It also must be a multiple of 3, since it is a multiple of 12, and of 7, since it is a multiple of 56. Hence

$$2^3 \cdot 3 \cdot 7 = 168$$

is a multiple of both 12 and 56, and by our choice it is the least positive multiple.

Recall that for the g.c.f. we took the lesser of the powers of the common primes, and notice that here for the l.c.m. we take the greater. In terms of the literal example of the previous section, suppose

$$m = p^a q^b r^c u^d, \qquad n = p^e q^f t^g ,$$

where p, q, r, u, t are distinct prime numbers. Then the l.c.m. of the two numbers must be divisible by p^a and p^e, so that if k is the greater of a and e, the l.c.m. will be a multiple of p^k. Similarly if s is the greater of b and f, the l.c.m. will be a multiple of p^s. Then the l.c.m. will be

$$p^k q^s r^c u^d t^g .$$

It may be enlightening to see what this discussion amounts to in terms of sets. Let M_6 be the set of positive multiples of 6 and M_{15} those of 15. Then

$$M_6 = \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96, \ldots\}$$

$$M_{15} = \{15, 30, 45, 60, 75, 90, 105, \ldots\}.$$

Then

$$M_6 \cap M_{15} = \{30, 60, 90, \ldots\}$$

represents the set of common multiples different from zero of 6 and 15. The least number in this set, 30, is the least common multiple of 6 and 15.

There is a slightly different point of view which exhibits somewhat more plainly the connection between the g.c.f. and the l.c.m., though it may not be as satisfactory for a class in junior high school. We return to the example of Section 4.4 and recall that b^0 is defined to be 1 for every number b which is different from zero. Then we can write the factorizations of 525 and 4455 as follows:

$$525 = 3 \cdot 5^2 \cdot 7 \cdot 11^0$$

$$4455 = 3^4 \cdot 5 \cdot 7^0 \cdot 11 ,$$

where $11^0 = 1 = 7^0$, and, of course, the lack of an expressed exponent means that it is 1. We have thus expressed each number as a product of powers of the same set of primes 3, 5, 7, 11 where now the exponents are whole numbers. To find the g.c.f. we use the product of these same primes but each time choose the smaller of the exponents in the two factorizations; that is, for 3 we choose the exponent 1, for 5 the exponent 1, and for 7 and 11 the exponents 0. We thus have

$$\text{g.c.f. of } 525 \text{ and } 4455 = 3 \cdot 5 \cdot 7^0 \cdot 11^0 = 15.$$

Now to get the l.c.m. we choose the larger of the two exponents for each prime factor, which is 4 for the prime 3, 2 for the prime 5, and 1 for 7 and 11. We thus have

$$\text{l.c.m. of } 525 \text{ and } 4455 = 3^4 \cdot 5^2 \cdot 7 \cdot 11 = 155{,}925.$$

The most common use of the l.c.m. is in the adding of two fractions. Though it is out of place in our logical development of the number system, the readers of this book are certainly familiar with fractions, and it is worthwhile to recall this connection. If we are to add

$$\frac{71}{525} \quad \text{and} \quad \frac{57}{4455} ,$$

we would express the sum in the following way:

$$\frac{71 \cdot 4455 + 57 \cdot 525}{525 \cdot 4455} = \frac{316{,}305 + 29{,}925}{2{,}338{,}875} = \frac{346{,}230}{2{,}338{,}875} .$$

But a more efficient way is to use the l.c.m. Here

$$\frac{\text{l.c.m.}}{525} = \frac{3^4 \cdot 5^2 \cdot 7 \cdot 11}{3 \cdot 5^2 \cdot 7} = 3^3 \cdot 11 = 297$$

and

$$\frac{\text{l.c.m.}}{4455} = \frac{3^4 \cdot 5^2 \cdot 7 \cdot 11}{3^4 \cdot 5 \cdot 11} = 5 \cdot 7 = 35 .$$

Then we can write equivalent fractions having the l.c.m. as the common denominator as follows:

$$\frac{71}{525} = \frac{71 \cdot 297}{525 \cdot 297} ; \qquad \frac{57}{4455} = \frac{57 \cdot 35}{4455 \cdot 35} .$$

Here the sum is equal to

$$\frac{71 \cdot 297 + 57 \cdot 35}{155{,}925} = \frac{21{,}087 + 1995}{155{,}925} = \frac{23{,}082}{155{,}925} .$$

In the former case, with larger numbers, there is a factor common to the numerator and denominator in the final fraction. In the latter case, as it happens, the final fraction is not in the simplest terms either, since the numerator and denominator have a common factor 3. The same methods apply when we are dealing with algebraic fractions. There is a slightly shorter method of performing the above, based on Problem 2 below.

Problems

1. We found in the previous section that for sets there was a close analogy between the intersection and the g.c.f. What is similarly analogous to the l.c.m. for two sets?

2. Notice that the product of 525 and 4455 is equal to the product of their g.c.f. and l.c.m. Is this an accident, or does this property hold in general? Why or why not? How may your result be used to shorten the process of summing the two fractions above?

3. What is the g.c.f. of the numerator and denominator of the fraction at the close of the section?

4. Bells A and B ring together at noon. If bell A rings every 12 minutes and bell B every 15 minutes, when will they next ring together?

5. Bell A rings at noon and bell B one minute after noon. If bell A rings every 12 minutes and bell B every 15 minutes, will they ever ring together, and if so, when is the first time?

6. If in the previous problem, 15 is replaced by 35, what would be your answer?

7. Can you generalize Problems 4, 5, and 6?

Exercises

1. Using the result of Problem 2 of this section and Exercise 2 of Section 4.4, find the l.c.m. of 17,325 and 407. Use this to express as a single fraction

$$\frac{79}{17,325} + \frac{11}{407}$$

2. Find the l.c.m. of the three numbers:

17,325, 407, 185

Can this be done without factoring any of them?

3. Let m be the l.c.m. of numbers a and b, and n the l.c.m. of m and c. Is n the l.c.m. of a, b, and c? (See Exercise 6 of the previous section.)

4. Let $[a,b]$ stand for the l.c.m. of a and b. Is this symbol associative, that is, is:

$$[[a,b],c] = [a,[b,c]]?$$

5. Show that if bc is the least common multiple of b and c , then their g.c.f. is 1. Is the converse of this statement true?

6. Given a pair of natural numbers, is there a least common divisor? Is there a greatest common multiple?

4.6 Tests for divisibility

Up to this point in this chapter we have been considering properties of the numbers themselves, irrespective of the particular system of numeration. In the decimal system 11 is a prime. In the numeral system to the base three it is written 102, but it still is a prime. In the numeral system to the base two it is written 1011, but it still is the fifth prime number. On the other hand, if we keep the numeral the same but change the numeral system, sometimes we have a prime number and sometimes not. In the numeral system to the base two 11 happens to be a prime because it is the prime number "three." But in the numeral system to the base five, 11 is not a prime number because it is the number "six"--in fact, in the base five $2 \cdot 3 = 11$.

In any numeral system there are ways of telling by looking at the numeral whether it has certain kinds of factors or not. In one system one can test certain numbers for divisibility quite easily, and for another system the easiest tests may be for numbers which are quite different. In this section we shall be concerned chiefly with tests for numbers expressed in the decimal system because it is here that these tests can be most useful.

The simplest kind of test is found by looking at the last digit. If the last digit of a number expressed in the decimal system is b , we can write the number as

$$10n + b$$

where n is a whole number. This means that if b is even, that is, divisible by 2, so is the number. That is, if b is one of 0, 2, 4, 6, 8 the number is even. Also if b is a multiple of 5, so is the number; that is, if b is one of 0, 5 the number is divisible by 5. But if the multiples of 3 less than 30 are listed, it is seen that a multiple of 3 can have any number as its final digit.

A second kind of test involves looking at all of the digits. The simplest of these for the decimal system is the test for divisibility by 9 or 3. We have

A test for divisibility by 9 (or 3). A number is divisible by 9 if and only if the sum of its digits is divisible by 9. In fact, we can prove a little more:

If a number is divided by 9, the remainder is the same as when the sum of its digits is divided by 9.

First of all, let us see why this is so for a particular number: 5348 for instance:

$$\begin{aligned} 5348 &= 5(1000) + 3(100) + 4(10) + 8 \\ &= 5(999 + 1) + 3(99 + 1) + 4(9 + 1) + 8. \end{aligned}$$

By the distributive property this is equal to

$$= 5(999) + 5 \cdot 1 + 3(99) + 3 \cdot 1 + 4(9) + 4 \cdot 1 + 8.$$

By the commutative property of addition we have

$$= 5(999) + 3(99) + 4(9) + 5 + 3 + 4 + 8.$$

This may be written:

$$5348 - (5 + 3 + 4 + 8) = 5 \cdot 999 + 3 \cdot 99 + 4 \cdot 9 .$$

By the distributive property the right side is a multiple of 9. Thus we have shown that the difference between 5348 and the sum of its digits is a multiple of 9. By Problem 4 in Section 4.2 this means that the remainders when the two numbers are divided by 9 are the same.

Just the same process works for divisibility by nine of any number expressed in the decimal numeral system. We illustrate this using letters for a four-digit numeral: abcd. (Here the juxtaposition of the letters does not indicate a product as it usually does, but merely that the digits of the number are a, b, c, d in order.)

$$\begin{aligned} abcd &= a(1000) + b(100) + c(10) + d \\ &= a \cdot 999 + b \cdot 99 + c \cdot 9 + a + b + c + d. \end{aligned}$$

Thus $$abcd - (a + b + c + d) = a \cdot 999 + b \cdot 99 + c \cdot 9.$$

Again the difference between the number and the sum of its digits is a multiple of 9, and hence the remainders when they are divided by 9 are the same.

This property used to be used for checking additions and multiplications by a process called "casting out the nines." We illustrate this for addition, but it can be used also for multiplication, division, and subtraction as well.

$$\begin{array}{rlr} 537 & \text{remainder after division by 9:} & 6 \\ \underline{4372} & \text{remainder after division by 9:} & \underline{7} \\ 4909 & & 13 \end{array}$$

Now 537 is a multiple of 9 plus 6: that is, $537 = 9 \cdot b + 6$ for some integer b. 4372 is a multiple of 9 plus 7: that is, $4372 = 9c + 7$ for some integer c. Thus

$$537 + 4372 = 9b + 6 + 9c + 7 = 9(b + c) + 6 + 7 .$$

This means that the remainder when the sum of the given numbers is divided by 9 is the same as the remainder when 6 + 7 = 13 is divided by 9. Now the sum of the numbers is 4909. The sum of its digits is 22, and the remainder when this is divided by 9 is 4 which is the same as the sum of the digits of the numeral 13.

Of course the process is shorter than the explanation. We merely add the digits of the sum and compare the remainder with that for the sum of the remainders. In fact, it is possible to shorten the process in various ways. To get the remainder when 22 is divided by 9, we need not divide but merely add the digits. Also, in dealing with 4909, we may omit the 9's (this is the origin of the name "casting out the nines") or, in fact, any sum divisible by 9 and merely write 4, the remaining digit.

This property of the number 9 with relation to our decimal system of numeration gave rise to a pseudo-science called numerology, in which you find the number 0 to 9 associated with your name according to the place of its letters in the alphabet and ascertain by looking in the book what your character is according to the number which comes from it.

It is important to emphasize that the above is a property definitely associated with the numeration system in which the number is written. For instance, in the numeration system to the base 6, we could not judge divisibility by 5 by looking at the last digit, since for instance, the number ten is in this system 14. But since 6 is 1 more than 5, we could test for divisibility by 5 in this system by adding the digits just as for divisibility by 9 in the decimal system. For example, in the numeral system to the base six, we can show that

1432

is divisible by 5. The sum could be computed either in the base 6 or the decimal system. For base 6 alone, however, one can continue to add digits as we did in the decimal system. That is, in the numeral system to the base six, the sum is 14 and the sum of its digits is 5. But in the decimal system the sum is 10, and the sum of its digits is not a multiple of five.

For us, tests for divisibility in other systems than the decimal system are not important. The only purpose in mentioning them at all is to emphasize the fundamental principles on which are based the tests for divisibility in the decimal system.

There are tests for divisibility by other numbers in the decimal system, but except perhaps for divisibility by 11, they seem to be of little practical use, especially in these days of computers.

Exercises

1. Find tests for divisibility by the following numers: 15, 45, 6, 8, and 72.

2. Use the results of Exercise 1 to test the following for divisibility by the numbers above:

   82710 , 67011 , 6228

3. In the numeral system to the base eleven, how would you test for divisibility by two?

4. For what numeral systems can one test for divisibility by two by merely looking at the last digit? For the other numeral systems how may one test for divisibility by two?

Problems

1. Establish the test for divisibility by 9 in the decimal system for the five-digit number abcde .

2. Find a test for divisibility by eleven in the decimal system. Hint: What are the remainders when the powers of 10 are divided by 11?

3. Show how the following trick can be performed and why it works. Also mention any possibility of failure. You select any number, form from it another number containing the same digits in a different order, subtract the smaller of the two numbers from the larger, and then tell me all but one of the digits of the result. I can then (perhaps) tell you what is the other digit of the result.

4. In the decimal system is there any other number besides 9 which could be tested for divisibility by adding the digits?

5. In the numeral system to the base six, what number would have a test for divisibility analogous to that for eleven in the decimal system?

6. In the numeral system to the base six, for divisibility by what numbers would looking at the last digit suffice, as for 5 and 2 in the decimal system?

4.7 Looking forward to the integers

Though we do not deal with negative numbers until later in this book, the reader who is familiar with them might be interested to see what modifications would be necessary in this chapter to include the negative integers as well. Actually very little change would be necessary. We would still require that any remainder be a whole number (not negative). Prime numbers, p, would have four instead of two factors: -1 and -p in addition to 1 and p. Not only 1 but also -1 would divide every number. The greatest common factor and least common multiple would still be whole numbers.

It would be quite a different matter if we were to consider the questions of this chapter for rational numbers, because in this set every number is divisible by every other number except zero. In fact, if r is a rational number different from zero, it can be expressed in infinitely many different ways as a product of rational numbers. To see this, we need merely note that if s is any rational number different from zero,

$$r = s\left(\frac{r}{s}\right)$$

and $\frac{r}{s}$ is a rational number. Hence there are no prime numbers in the set of rational numbers.

4.8 Some properties of prime numbers

In this section we deal with some of the properties of prime numbers which may be of interest to the reader. (This section is not essential to what comes later.) First we show how we can prove the Fundamental Theorem of Arithmetic assuming only results derived from the euclidean algorithm. Recall

Theorem 1. Every counting number can be expressed as a product of prime numbers, and the product is unique except for the order of the factors.

We proved the first part of this theorem in Section 4.3. For the second part our chief tool is Theorem 3 of Section 4.4. We repeat it here for easy reference:

> If a is a factor of bd and if a and b have g.c.f. 1, then a is a factor of d .

Now then, for the proof, suppose n is expressed in two different ways as a product of prime factors as follows:

$$n = p_1p_2p_3 \cdots p_r = q_1q_2q_3 \cdots q_s ,$$

where the p's and q's are prime numbers, not necessarily distinct.

We want to prove that $r = s$ and that the following sets are equal:

$$\{p_1, p_2, p_3, \dots, p_r\}, \qquad \{q_1, q_2, q_3, \dots, q_s\}$$

where any repeated prime in the first set occurs the same number of times in the second set.

Suppose first, for convenience, we order the factors so that in each product the primes occur in order of size, that is:

$$p_1 \le p_2 \le p_3 \le \cdots \le p_r, \qquad q_1 \le q_2 \le q_3 \le \cdots \le q_s .$$

First we want to show that $p_1 = q_1$. Suppose p_1 were smaller than q_1. Then since p_1 is a prime number, the g.c.f. of p_1 and q_1 would have to be 1. But p_1 is a factor of q_1c where c is the product of all but the first q. Thus by Theorem 3 quoted above, p_1 must be a factor of c. Now since p_1 is less than q_1, it is certainly less than q_2, and by the same argument we see that p_1 must be a factor of the product

$$q_3q_4q_5 \cdots q_s .$$

Continuing this argument, we see that p_1 would have to be a factor of the product of the last $s - 3$ primes q, then of the product of the last $s - 4$ primes q, and so forth. In the end p_1 would have to be a factor of q_s which is impossible since $q_s > p_1$. This was all on the assumption that p_1 were less than q_1. If q_1 were less than p_1, we would repeat the same argument with p and q interchanged. Thus we have a contradiction unless $p_1 = q_1$.

So now we have

$$p_1p_2p_3 \cdots p_r = q_1q_2q_3 \cdots q_s$$

with $p_1 = q_1$. We may then divide both sides by p_1 and have

$$p_2p_3p_4 \cdots p_r = q_2q_3q_4 \cdots q_s .$$

Each of these products has one less term than before, and we can repeat the whole argument. We can continue to repeat the argument until either the p's or the q's are exhausted. Then it follows that $r = s$ and the p's and q's are equal in pairs. This is what we wished to prove.

Another important result about prime numbers is the following:

Theorem 2. The number of prime numbers is infinite; that is, no matter what number you name, there is more than that number of primes. (This proof was known in Euclid's time and is generally attributed to him.)

To prove this, we assume that the statement is false; that is, some number n represents the number of different primes. If that were the case, we could indicate the set of primes by

$$p_1, p_2, p_3, \cdots, p_n$$

where p_1 is the first prime, namely 2, p_2 the second prime, namely 3, and so forth until we get to p_n which is the last prime. Let P be the product of these prime numbers and write the number

$$N = P + 1 .$$

Now N is not divisible by p_1 because the remainder when you divide N by this number is 1; it is not divisible by p_2 for the same reason; in fact, it is not divisible by any of the primes we have listed. In fact, the remainder is 1 when N is divided by any factor of P. So it must be either a prime itself, different from all the others, or it must have a prime factor which is different from all those listed. In either case we have shown the existence of one more prime than we started with. This is the contradiction we sought.

The reader should be warned about two things in connection with this result. In the first case, N need not be the $(n + 1)$th prime. In fact, there is a result which tells us that, except for $n = 1$, N cannot be the next prime. (We refer to this result in a problem below.) Secondly, N need not be a prime.

One of the difficulties and attractions of prime numbers is their irregular distribution. The primes 2 and 3 are the closest together, and the minimum difference between any two other primes is 2 because every second number is divisible by 2. Thus after 2 and 3, the primes which are the closest together differ by 2. These are called "twin primes" and the first five pairs are:

$$3, 5; \;\; 5, 7; \;\; 11, 13; \;\; 17, 19; \;\; 29, 31 .$$

It is not known whether or not there is an infinite number of such pairs.

On the other hand, two successive prime numbers can be as far apart as you please. For instance, suppose you want to see ten consecutive numbers none of which is a prime number. For this let

$$n = 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11$$

that is, the product of the first ten natural numbers after 1. (A short way of writing this number is 11!.) Then form the following set of ten consecutive whole numbers:

$$n + 2, n + 3, n + 4, n + 5, n + 6, n + 7, n + 8, n + 9, n + 10, n + 11.$$

The first of these is divisible by 2 since n is, the second is divisible by 3 because n is, the third by 4 because n is, ..., and the last divisible by 11 because n is. Now n + 12 might or might not be a prime number. As it happens, n + 12 is not a prime number, since n is divisible by 3. But n + 13 might or might not be a prime number. In any case, we have shown that there are two prime numbers whose difference is greater than 10. Similarly, we could exhibit 90 consecutive integers, all composite, or any other number of consecutive composite integers.

Problems

1. There is a theorem, much too difficult to prove here, which is called "Bertrand's Postulate" and which tells us that between every number greater than 1 and its double, there is at least one prime number. For instance, between 5 and 10 there is the prime 7; between 11 and 22 there is the prime number 13 (as well as 17 and 19). Use this result to prove that the number N in our proof above of the infinitude of primes can never be the (n + 1)th prime number unless n = 1.

2. In the last paragraph of this section we exhibited ten consecutive numbers, all of which are composite. A somewhat more efficient choice of n would have been to choose it to be the product of the prime numbers less than or equal to 11, that is

   $$n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2,310 ,$$

   and then write the following set of ten consecutive whole numbers:

   n + 2, n + 3, n + 4, n + 5, n + 6, n + 7, n + 8, n + 9, n + 10, n + 11.

   Why are all of these composite numbers? Is n + 12 composite? Why is this somewhat more efficient?

4.9 Other systems

The fundamental theorem of arithmetic does not hold in all systems. A simple example of such a system is the following:

$$1, 2, 4, 6, 8, 10, \ldots,$$

that is, the set of even counting numbers and the number 1. This set is closed under multiplication (though not under addition). A prime number we would naturally define as a number of the set, different from 1, which could not be represented as a product of two numbers of the set neither of which is 1. With this definition it is easy to see that the prime numbers for this set are the even numbers not divisible by 4. Now 60 can be written in two different ways as a product of prime numbers of this set: $60 = 2 \cdot 30 = 6 \cdot 10$. These products do not contain the same prime numbers, and hence the fundamental theorem of arithmetic does not hold here.

On the other hand, there are many systems besides the counting numbers for which the fundamental theorem of arithmetic does hold.

For instance, let D be the set of numbers of the form $\frac{a}{2^k}$ where a is an integer and k is a whole number. This set is closed under addition and multiplication. Here we must introduce a new term, that of a unit. In the set of whole numbers, 1 and -1 are peculiar in that they are divisors of every number; that is, their reciprocals are integers. In general, a number of a set S is said to be a unit if its reciprocal (multiplicative inverse) is in S. In the set of rational numbers every number except zero is a unit. In our set D, the number $\frac{a}{2^t}$ is a unit if and only if $\frac{2^t}{a}$ is also in D. But if $\frac{2^t}{a}$ is to be in D, then a must be a power of 2, say 2^r. Thus the units of D are the numbers: $\frac{2^r}{2^t} = 2^{r-t}$ where $r-t$ is an integer which may be positive, negative, or zero. Thus the units of D are:

$$\frac{1}{2^w}, \quad \frac{-1}{2^w}, \quad 2^w, \quad -2^w$$

where w is a whole number. (Actually, since w might be negative, we could omit the first two forms.) Now by analogy with the set of integers, we call a number m in D a prime number in D if it is not a unit and if

$$m = ab, \quad \text{with } a \text{ and } b \text{ in } D,$$

implies that a or b is a unit.

To find the prime numbers of D, notice first that if m is an element of D, it may be written

$$m = un,$$

where u is a unit of D and n is an odd integer. This is true, since we can take u as the highest power of 2 in m if m is an integer, or the denominator of m if m is not an integer. Hence to find the prime numbers of D, we need only find those among the odd integers.

First, if n is not a prime number, it can be expressed in the form $n = ab$ where both a and b are different from 1 and -1. Then a and b, being integers, are in D, neither is a unit, both are odd, and hence n is not a prime number in D. Conversely, suppose n is not a prime number in D; then

$$n = \frac{c}{2^r} \cdot \frac{d}{2^s}$$

where neither fraction is a unit. Assuming each of the fractions to be in simplest form, we see that c and d must be odd and hence n being an integer implies that r and s are both zero. Thus n = cd with neither c nor d a unit, that is, neither is 1 nor -1, and n is not a prime number. Hence we have shown that the prime numbers of D are of the form

$$pu$$

where p is an odd prime number and u is a unit of D.

Problems

1. Let S be the system of numbers described in the first paragraph of this section. Does the following result hold? If a and b are two numbers of S , then there are numbers q and r of the set S such that

$$a = bq + r, \quad 0 \leq r < b \ .$$

2. Consider the following set of numbers: $\frac{a}{b}$ where b is an odd integer and a is an integer. What are the units for this set? What are the primes? Does the Fundamental Theorem of Arithmetic hold for this set?

References

4 (Numerology), 9 (Chapter 1), 12 (Chapter 4), 13 (Chapter 1, Appendices A and B), 14, 17, 20 (Chapters 2, 3, 4), 21 (Chapter 1), 22 (Chapter 4)

Chapter 5

THE NON-NEGATIVE RATIONAL NUMBERS

5.1 Introduction

As we noted at the beginning of the previous chapter, the set of whole numbers is not closed with respect to division. The purpose of the introduction of rational numbers is to remedy this deficiency. We want to preserve as many as possible of the properties we found so useful in the manipulation of whole numbers. At the same time we need to have a system which fits in with our everyday needs. Luckily it turns out that if we are willing to let go by the board most of the properties in the previous chapter, which were really forced on us by the nature of the system we were considering, we can have both a useful system and one that has the properties we described for the whole numbers in Chapter III.

The use for fractions needs no justification, but the need for understanding them again rests not only on the rules for manipulation but on the reasons for setting up the rules in the first place. Of course one does the manipulations by rote just as one memorizes the multiplication table. But the reasons put a firm basis under the manipulations as well as enlighten what went before. Furthermore, for those going on to algebra, here in a simpler setting are reasons and processes which recur later in the less familiar setting of algebra.

The plan of this chapter is as follows. We define what we mean by a rational number and give the conditions for equality so that important properties will hold. Addition and subtraction, multiplication and division are defined from the same point of view. This we do in detail for the reasons mentioned above. It is also important to check along the way to be sure that these new numbers do what we want them to do; that is that they are a model for the applications we have in mind. The teacher may not want to go into as much detail in class as we do here but certainly the pupil has a right to know why definitions are set up as they are.

The second step would be to verify that indeed these numbers have the properties with which we were familiar in Chapter III. Since this is not a complete treatise in this sense, we give only a sample of what would have to be done to be complete.

The third step would be to acquire dexterity in the handling of the processes. This the student presumably already has to a certain extent, but undoubtedly he needs more. Since the acquisition of this facility is primarily a matter of practice, the author feels that he can contribute little to this important topic, except by the inclusion of sample exercises in this text.

Since fractions seem easier than negative numbers we deal with positive rational numbers in this chapter and leave negative integers and rational numbers to a later point in the book.

5.2 The definition of rational numbers

The mathematical need for an extension of our number system and the practical need extend from the same problem--the need to divide something into a number of equal parts. If we are to divide a pie into five equal parts we must, in mathematical terms, solve the equation $5x = 1$. In fact we need a number for a solution of any equation of the form $ax = 1$ provided that a is any number different from zero. So, taking our cue from the notation for quotient in Chapter IV, we create a whole new set of numbers and denote by

$$1/a \quad \text{or} \quad \frac{1}{a} \quad \text{or} \quad a^{-1}$$

the solution of $ax = 1$ with $a \neq 0$. We exclude $a = 0$ because $0 \cdot x = 0$. More about this later.

Physically this number $\frac{1}{a}$ indicates cutting something up into a equal parts. In other words, it has the property that if you multiply it by a you get 1. We could indicate this on a line as follows, where a is taken to be 8.

The distance between each pair of dots is one-eighth of the whole.

To be useful this new set of numbers should have certain properties. For one thing, it should have the associative property for multiplication. In this case:

$$a[(\frac{1}{a})b] = [a(\frac{1}{a})]b = 1 \cdot b = b, \; a \neq 0,$$

and we see that for $x = (\frac{1}{a})b$, it is true that $ax = b$. Also we would want these new numbers to be associative and commutative under multiplication with themselves and the integers. Thus we have

$$[(\frac{1}{a})b]a = [b(\frac{1}{a})]a = b[(\frac{1}{a})a] = b.$$

Hence $x = (\frac{1}{a})b$ is a solution of both the equations: $ax = b$ and $xa = b$. Then we designate $b(\frac{1}{a})$ and $(\frac{1}{a})b$ by the symbol $\frac{b}{a}$ and create a set of numbers $\frac{b}{a}$ where b and a are whole numbers with a different from zero and having the following property:

$$a(\frac{b}{a}) = (\frac{b}{a})a = b$$

for b a whole number and a a counting number. That is, our new number $\frac{b}{a}$ will be a solution of the equations $ax = b$ and $xa = b$. Formally, we have the following:

Definition: Numbers which can be represented by fractions b/a or $\frac{b}{a}$ where b is a whole number and a a counting number are called rational numbers. In other words, the rational numbers are the solutions of equations: $ax = b$ where b is a whole number and a a counting number. (This definition will be extended later to cases when b and a may be negative.)

This creation raises a number of questions. Perhaps the first is, how do we know that such numbers exist? The answer to this question is that we know they exist because we bring them into being. A harder question is: why is there only one solution to the equation $ax = 1$? If there were two, x and y, we would have $ax = 1 = ay$. Then if the cancellation property is to hold, the equation $ax = ay$, with $a \neq 0$, would imply $x = y$.

Of course, a larger question is: will these numbers do what we want them to do? Here we have to take some initiative and so manage our definitions that these numbers are our faithful well behaved servants. Also there goes with it a certain amount of faith that our definitions will not lead us into trouble. Luckily we shall find that at each stage there is really only one choice to make if these new numbers are to behave as we wish them to.

We have already made the choice that $b(\frac{1}{a})$ shall be equal to $(\frac{1}{a})b$ by writing them both $\frac{b}{a}$. This checks with our practical experience for we know that if we take one-seventh of a pie and consider two of these pieces, we will have the same amount of pie as if we had found one-seventh of two pies.

As a matter of fact, parenthetically, our practical experience keeps us straight many times in mathematics. We assume the world about us to be consistent and if we make our mathematical system a model of the world as we know it we can feel reasonably sure of consistency in the mathematics. Higher mathematics often leaves the ordinary world far behind or, from another point of view, becomes an expansion itself of the world. Then new instruments for navigation are necessary. But here we are close to our known world and we must maintain contact with it, for our own comfort and security.

5.3 Equality of rational numbers

Above, by definition, we made the product of a whole number and a rational number commutative. Here we want to make all such products associative. The solutions of

$$ax = b \quad \text{and} \quad cax = cb$$

should be the same since we want $c(ax)$ to be equal to $(ca)x$. But, by our definition, the solution of the first equation is $\frac{b}{a}$ and that of the second is $\frac{(cb)}{(ca)}$. Thus we agree that

$$\frac{b}{a} = \frac{cb}{ca}$$

is to be true whenever a, b, c are counting numbers. This is important enough to state formally as:

Property 1. If the numerator and denominator of a fraction are multiplied or divided by the same counting number (or as we shall later see by any number different from zero) the number represented is not changed.

We can check this with experience. Suppose a line segment is marked off in twelve equal divisions. One third of the segment will be four twelfths:

$$\frac{1}{3} = \frac{4}{12}.$$

In general, if a line segment is divided into ba equal segments, then b of these segments combined will have a length equal to $\frac{1}{a}$ of the given segment. That is, $\frac{1}{a} = \frac{b}{ba}$.

Here we need a word to describe the relationship between two fractions which represent the same number. Traditionally two such fractions are called equal, but for reasons discussed in the next section we prefer to call them, at least for the time being, "equivalent." Thus:

Definition: Two fractions are said to be equivalent if they represent the same number.

Property 1 does not tell the complete story as far as equivalence of fractions is concerned. For instance, consider

$$\frac{4}{10} \quad \text{and} \quad \frac{6}{15}.$$

By Property 1, $\frac{4}{10}$ and $\frac{2}{5}$ are equivalent fractions and so are $\frac{6}{15}$ and $\frac{2}{5}$. All three of $\frac{4}{10}$, $\frac{2}{5}$ and $\frac{6}{15}$ represent the same number. Thus $\frac{4}{10}$ and $\frac{6}{15}$ are equivalent even though one cannot get $\frac{6}{15}$ from $\frac{4}{10}$ by multiplying or dividing the numerator and denominator by a counting number.

There are two ways of determining whether two fractions are equivalent. Probably the simpler as well as the less familiar is given in

Theorem 1: Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with b and d different from zero are equivalent if and only if $ad = bc$.

The proof is not hard. First suppose $ad = bc$. Then $\frac{a}{b}$ and $\frac{ad}{bd}$ are equivalent by Property 1, $ad = bc$ implies $\frac{ad}{bd}$ and $\frac{bc}{bd}$ are equivalent and, by Property 1, $\frac{bc}{bd}$ and $\frac{c}{d}$ are equivalent. Thus $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent. Conversely, suppose $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent. By Property 1, $\frac{ad}{bd}$ and $\frac{a}{b}$ are equivalent; for the same reason, $\frac{bc}{bd}$ and $\frac{c}{d}$ are equivalent. Thus $\frac{bc}{bd}$ and $\frac{ad}{bd}$ represent the same number. That is, $bc\left(\frac{1}{bd}\right)$ and $ad\left(\frac{1}{bd}\right)$ represent the same number, which implies $bc\left(\frac{1}{bd}\right)bd = ad\left(\frac{1}{bd}\right)bd$ or $bc = ad$, assuming the associative property.

This method of determining the equivalence of two fractions has the advantage that it merely requires the comparison of two products and also it holds not only for counting numbers but for any numbers discussed in this book.

The second method of determination is somewhat more laborious but, since it is in common practice and has some advantages over the above we discuss it too. To this end we first define a fraction "in simplest form" or "in lowest terms."

Definition: A fraction $\frac{a}{b}$ is said to be in simplest form or in lowest terms if a and b are counting numbers without a common factor greater than 1.

This definition is useful because every rational number can be represented by a fraction in lowest terms. To see this merely note that if g is the g.c.f. of c and d, we may divide numerator and denominator of the fraction $\frac{c}{d}$ by g and have one in simplest form since, by Section 4.4, Theorem 2, $\frac{c}{g}$ and $\frac{d}{g}$ have g.c.f. 1. For example $\frac{4}{10}$ and $\frac{6}{15}$ are both equivalent to $\frac{2}{5}$ in simplest form.

Then we have the following:

Theorem 2: Two fractions are equivalent if and only if their simplest forms are the same. (See the discussion of the properties of an equivalence relation in the next section.)

To prove this, note that if they are equivalent to the same fraction they are equivalent to each other. To complete the proof we need merely show that two fractions in simplest form are equivalent only if they are the same. For this, suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two equivalent fractions in simplest form. Then, from Theorem 1, we have $ad = bc$. Since $\frac{a}{b}$ is in simplest form, 1 is the g.c.f. of a and b; hence, from Section 4.4, a is a factor of c. Similarly, since $\frac{c}{d}$ is in simplest form and c is a factor of ad, c is a factor of a. Thus c and a are factors of each other and, by Section 4.2, $a = c$. Similarly, $b = d$ and our proof is complete.

We want every counting number to be a rational number. For instance, $2x = 6$ has the solution 3 and also, by our definition of fractions, the solution $\frac{6}{2}$, which is equivalent to the fraction $\frac{3}{1}$. So we agree that $\frac{3}{1}$ and 3 shall be two ways of representing the same number. In general $1 \cdot x = a$ has the solution $x = a$ and also $\frac{a}{1}$. So by definition

$$\frac{a}{1} = a$$

for every counting number.

Finally, we have avoided fractions with numerator zero. This was just a matter of convenience. The fraction $\frac{0}{a}$ should be the solution of $ax = 0$. We know that this has the solution 0 and hence we define

$$\frac{0}{a} = 0, \text{ when } a \neq 0.$$

This is quite a different matter from having the denominator zero. Suppose for instance that $x = \frac{1}{0}$ were a number. This would have to mean $1 = 0 \cdot x$. If we were to multiply both sides by 2 and assume the associative property we would have

$$2 = 2 \cdot 1 = 2 \cdot (0 \cdot x) = (2 \cdot 0) \cdot x = 0 \cdot x = 1.$$

This would mean that we would either have to accept the equality $2 = 1$ or dispense with the associative property. Hence we outlaw division by 0 and fractions with zero denominator.

Problems

1. Prove that two fractions with the same denominator represent the same rational number if and only if the numerators are the same.
2. Does the conclusion of the previous problem hold if the words "numerator" and "denominator" are interchanged?

5.4 "Simplest" or "canonical" forms

At this point we digress from our development of the rational number system to look a little at what we have done. Previous to this chapter we had various ways of representing numbers. For instance the following all represent the same number

$$2 + 3,\ 7 - 2,\ 1 + 1 + 1 + 1 + 1,\ \frac{10}{2}.$$

They are all representations of the number five. We would probably agree that the simplest way to represent the number five is by the single numeral 5. The Mayans would not have agreed with this, they would have used simply a bar: ____ . One way to show that two expressions represent the same number is to show that they represent the same "simplest" number in some well-defined sense. Though we may differ on what is "simplest," and indeed this may well depend on the use we are going to make of it, yet it should have two fundamental properties:

1. Every number of the kind we are considering should be expressible in our "simplest form."

2. For a given number there is only one "simplest form," that is, the "simplest forms" must look the same.

The mathematical term for such a form is "canonical form." Though it is not in the least necessary to use this term, the concept is an important one. For instance, if you were asked, "What is the sum of $\frac{1}{3}$ and $\frac{2}{5}$?" a perfectly proper answer would be

$$\frac{1}{3} + \frac{2}{5}.$$

This is a representation of the number which is the sum of the two given rational numbers. But to find the simplest form which represents the sum of the numbers, then some calculation needs to be performed. It is the opinion of the author that some of the trouble which teachers have with students is because of the lack of giving definite instructions. Instead of asking, find the sum of $\frac{1}{3}$ and $\frac{2}{5}$, the request should be something like: express as a single fraction in simplest form the sum: $\frac{1}{3} + \frac{2}{5}$.

Here also are questions of numerals versus numbers. We have here considered a fraction to be a numeral--something which represents a number. We use the term "equivalence" when referring to two fractions first of all to emphasize the above ideas and second to make a bow to what we hope will become the tradition in geometry and elsewhere of calling things equal only if they are the same, that is, identical. Thus if we are to express ourselves in accordance with a strict use of the term "equal" we would call two fractions

$\frac{a}{b}$ and $\frac{c}{d}$ equal only if they are the same, that is $a = c$ and $b = d$; they are equivalent if $ad = bc$. We have tried to be careful in our language to be consistent with this concept. It is easy to slip and if carried to extremes, this care can be ridiculous pedantry. For instance, what is the numerator of a fraction? Is it a number or a numeral? Since it is part of a numeral presumably it has to be a numeral itself. So we do not really multiply the numerator of a fraction by a counting number; we multiply by a counting number the number which the numerator represents, and then we write the numeral which represents this product in forming the new fraction. This is of course a ridiculous way to express ourselves. In the opinion of the author there is really no harm in calling two fractions equal when they represent the same number, provided that this double meaning is pointed out explicitly.

In various connections we shall meet this idea of an equivalence relation and hence at this point it is well to consider what properties it has. Consider some set containing elements a, b, c, ... These elements may be numbers, triangles, equations--what you will. Suppose there is a relation between any two elements of the set and call it R. If the set is a set of numbers, R might be equality. If the set is a set of triangles, R might be congruence. If the set is a set of persons, R might be the relation: "live in the same house as." Whatever it is, we call R an equivalence relation if it has the following three properties.

1. (reflexive) $a R a$ for all a in the set.
2. (symmetric) If $a R b$ then $b R a$.
3. (transitive) If $a R b$ and $b R c$, then $a R c$.

Let us try this out on the examples given above. Equality is certainly an equivalence relationship because a number is equal to itself, if $a = b$ then $b = a$ and, finally, if $a = b$ and $b = c$ then $a = c$. The last in Euclidean phraseology is "two things equal to the same thing are equal to each other." Congruence of triangles is also an equivalence relationship. As for the third example notice first that a person lives in the same house as himself, next if person A lives in the same house as B then B lives in the same house as A, and finally, if A and B live in the same house and B and C live in the same house, then certainly A and C live in the same house.

However, "is the brother of" is not an equivalence relationship among people for it satisfies none of the properties above. The relationship among people "is not taller than" satisfies Properties 1 and 3 but not 2.

An equivalence relation always leads to a classification, provided we put in the same class all elements which are equivalent to each other. It has the following properties analogous to those for an equivalence relation:

1c. An element A is in its own class.

2c. If A and B are in the same class so are B and A.

3c. If A is in the same class as B and B in the same class as C, then A and C are in the same class.

In fact, being in the same class is then an equivalence relation. In the first example above we put into one class all the numbers equal to a given one, in the second case we put all triangles congruent to one in the same class, and in the third case we classify people by the houses they live in.

In the light of this general discussion, let us return to the idea of equivalence of fractions and write in two parallel columns corresponding statements about fractions and rational numbers. To simplify this writing, we denote "the fraction $\frac{a}{b}$ is equivalent to $\frac{c}{d}$" by: $\frac{a}{b} \sim \frac{c}{d}$. By $\frac{a}{b} = \frac{c}{d}$ we mean that the numbers which the fractions represent are equal.

	Fractions	Rational Numbers
Property	$\frac{a}{b} \sim \frac{c}{d}$	$\frac{a}{b} = \frac{c}{d}$
Reflexive	$\frac{a}{b} \sim \frac{a}{b}$	$\frac{a}{b} = \frac{a}{b}$
Symmetric	If $\frac{a}{b} \sim \frac{c}{d}$ then $\frac{c}{d} \sim \frac{a}{b}$	If $\frac{a}{b} = \frac{c}{d}$ then $\frac{c}{d} = \frac{a}{b}$
Transitive	If $\frac{a}{b} \sim \frac{c}{d}$ and $\frac{c}{d} \sim \frac{e}{f}$ then $\frac{a}{b} \sim \frac{e}{f}$	If $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$ then $\frac{a}{b} = \frac{e}{f}$

The parallelism results from the fact that both equivalence in this sense and equality are equivalence relationships. We also have a classification of fractions obtained by putting in the same class all fractions which are equivalent to a given fraction, that is, all those which represent the same number.

Problems

1. Show that the equivalence of fractions satisfies the three properties above of an equivalence relation.
2. Show that if there is a classification satisfying the three properties 1c, 2c, 3c then there is a corresponding equivalence relation.
3. Find examples of relationships which satisfy two of the properties of an equivalence relation but not the third.

5.5 Multiplication of rational numbers

Since multiplication is here easier than addition, we consider it first. Let $x = \frac{b}{a}$ and $y = \frac{d}{c}$. These mean that $ax = b$ and $cy = d$. Thus

$$bd = (ax)(cy) = (ac)(xy)$$

where for the first equality we made use of the fact that ax and b represent the same number and also cy and d. For the second equality we assumed that our rational numbers have the associative and commutative properties for multiplication. But $bd = (ac)(xy)$ means $xy = \frac{bd}{ac}$. Thus we have

$$\frac{b}{a} \cdot \frac{d}{c} = \frac{bd}{ac} .$$

In other words, we have the

Definition for multiplication: The product of two numbers in fractional form is represented by the fraction whose numerator is the product of the numerators of the fractions and whose denominator is the product of the denominators. That is, to find the product of two numbers in fractional form, multiply the numerators and divide by the product of the denominators.

Fortunately this checks with some other properties of fractions which are very useful. Returning to pies, we know that one-half of one-third of a pie is one-sixth. This checks with our multiplication procedure above:

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6} .$$

Furthermore: $2 \cdot 3 = \frac{2}{1} \cdot \frac{3}{1} = \frac{6}{1} = 6$ which is also fortunate. Finally:

$$\frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{r}{r} \cdot \frac{a}{b} = \frac{ra}{rb}$$

for counting numbers a, b and r. Thus our definition of multiplication is consistent with the property of equivalence of fractions.

Exercises

1. Prove that if $\frac{a}{b} = \frac{c}{d}$ and $\frac{r}{s} = \frac{t}{u}$, then

$$(\frac{a}{b})(\frac{r}{s}) = (\frac{c}{d})(\frac{t}{u}).$$

2. Let R be an equivalence relation for a set of numbers S and suppose that if a, b and c are in S, then a R b implies ac R bc. Then prove that if a R b and c R d, then ac R bd.

3. In his will, Farmer Brown specified that one-half of his estate should go to his eldest, one-third to the next in line and one-ninth to his youngest. In due course he died, whereupon the three sons set out to divide the estate in accordance with their father's wishes. This was easy for the land and they sold the house. But when it came to the 17 horses, no agreement could be reached and the more they talked the more bitter the argument became. Their neighbor, hearing the altercation, came to see what the difficulty was. He said that in the interest of peace among neighbors, he would contribute his horse. That made all things easy for with 18 horses, the eldest received his share, 9 horses, the second his share, 6 horses and the youngest his, 2. As it happened, the neighbor's horse was left over and he took it back home. What is the moral of this tale?

Problem

1. Show that if we define $(\frac{b}{a})(\frac{d}{c}) = (\frac{bd}{ac})$ when one or both of b and d are zero, this is consistent with our definition of $\frac{0}{c} = 0$ for $c \neq 0$.

5.6 Division of rational numbers

We devised rational numbers so that we could divide the undivided. We seem to have accomplished it for counting numbers in that we can solve $ax = b$ for counting numbers a and b. But what happens if a and b are themselves rational numbers? We take care of this problem in two parts.

First we wish to find a solution of the equation:

$$\frac{c}{d} x = 1.$$

Since we want the well-defined property for multiplication to hold, the solution x should satisfy the equation:

$$d\left(\frac{c}{d}x\right) = d.$$

From the associative property this implies

$$\left(d \cdot \frac{c}{d}\right)x = d.$$

But $\left(d\,\frac{c}{d}\right) = c$ and hence the equation above becomes

$$cx = d.$$

Thus x must be equal to $\frac{d}{c}$ and we have shown that if the properties we desire hold, the solution of

$$\frac{c}{d}x = 1$$

must be $\frac{d}{c}$. That this is indeed a solution can be verified as follows:

$$\frac{c}{d} \cdot \frac{d}{c} = \frac{cd}{dc} = 1.$$

Mechanically then the fraction which represents the solution of the equation

$$\frac{c}{d}x = 1$$

is obtained by interchanging the numerator and denominator of $\frac{c}{d}$.

The numbers $\frac{a}{b}$ and $\frac{b}{a}$ are called *reciprocals* or *multiplicative inverses* of each other. Their product is 1, as may be verified directly by the definition of multiplication of fractions. In the set of whole numbers there was only one number which had a reciprocal in the set: the number 1. If we were to look ahead and include the negative integers as well, we would have only one more: -1. But for the rational numbers we have shown that every number except zero has a reciprocal. This is very convenient for if we wish to solve the equation $rx = s$ where r and s are rational numbers we need merely multiply both sides of the equation by the reciprocal of r and use the associative property of multiplication as follows:

$$rx = s \text{ implies } \frac{1}{r}\,rx = \frac{1}{r}\,s\,.$$

But $\left(\frac{1}{r}\right)rx = \left(\frac{1}{r}\right)r \cdot x = s$ and we see that the solution of $rx = s$ is

$$x = \left(\frac{1}{r}\right)s.$$

By the definition of $\frac{s}{r}$ and the definition in Section 5.5 of the product of two numbers in fractional form,

$$\left(\frac{1}{r}\right)s = \left(\frac{1}{r}\right)\left(\frac{s}{1}\right) = \frac{s}{r}\,.$$

If we express r and s as fractions $\frac{a}{b}$ and $\frac{c}{d}$, we have

$$\frac{a}{b}x = \frac{c}{d}.$$

If we multiply both sides by the reciprocal of $\frac{a}{b}$ we have

$$(\frac{b}{a})(\frac{c}{d}) = (\frac{b}{a})(\frac{a}{b})x = 1 \cdot x = x.$$

Then $x = \frac{bc}{ad}$ is the solution of $(\frac{a}{b})x = (\frac{c}{d})$.

By showing how to solve the equation $rx = s$ with r and s rational numbers, we have shown that the set of rational numbers excluding zero is closed under division.

Exercises

1. By definition, the solution of $ax = b$ is $\frac{b}{a}$ for whole numbers b and a, with a different from zero. Why does this also hold when a and b are rational numbers, with a different from zero?

2. Let $x = \frac{a}{b}$, $y = \frac{c}{d}$ and $z = \frac{e}{f}$ where a, b, c, d, e and f are whole numbers with none of b, d, f zero. Find expressions as a single fraction for each of the following:

   a) $(x/y)/z$ b) $x/(y/z)$

   Are the two results equal?

3. Is $\frac{1}{(a/b) + (c/d)} = \frac{b}{a} + \frac{d}{c}$ assuming that all letters stand for whole numbers and no denominator is zero? (Hint: check this first for numbers.)

4. For rational numbers x, y, and z (none zero) does the following hold:

   $$(x/y)/z = z/(y/x)?$$

   (Hint: check this first for numbers).

Problem

1. Verify directly that $(\frac{a}{b})x = (\frac{c}{d})$ has the solution $\frac{bc}{ad}$.

5.7 Addition of rational numbers

Here, as above, there are two problems. First, how can we define addition of rational numbers so that it will have the properties we want? Second, after we know the answer to the first, what is the process or *a* process by which we can express as one fraction the sum of two numbers represented by fractions?

For the first, we can proceed as in the section on multiplication. We have $ax = b$ and $cy = d$ and wish to find $x + y$. If we are to use the distributive property, that is, if rational numbers are to have this property we need to change the given equations into equivalent ones in which the coefficients of x and y are the same. We can do this by replacing

$$ax = b \text{ by } cax = bc \text{ and } cy = d \text{ by } acy = ad.$$

Then we have

$$bc + ad = cax + acy = ac(x + y).$$

This means that $x + y$ must be $\frac{(bc + ad)}{ac}$. In other words we have the following definition:

$$\frac{b}{a} + \frac{d}{c} = \frac{bc + ad}{ac}.$$

We have thus found the single fraction which represents the sum of the numbers indicated on the left side of the equality. The fraction $\frac{(bc + ad)}{ac}$ may not be in simplest form, but it certainly is a single fraction which represents the sum.

There is another way of arriving at the same result. This stems from the observation that: *If two fractions have the same denominator a fraction representing the sum is one which has the same denominator as the two and whose numerator is the sum of the numerators of the two.* This, again, depends on our desire to have the distributive property. Let us see how this goes:

$$\frac{r}{s} + \frac{t}{s} = r\left(\frac{1}{s}\right) + t\left(\frac{1}{s}\right).$$

If the distributive property is to hold, $r\left(\frac{1}{s}\right) + t\left(\frac{1}{s}\right)$ must be equal to

$$(r + t)\left(\frac{1}{s}\right) = \frac{r + t}{s}.$$

This proves the underlined statement above.

So now, if we have fractions $\frac{b}{a}$ and $\frac{d}{c}$ we first write them as two equivalent fractions with the same denominator:

$$\frac{b}{a} = \frac{bc}{ac} \text{ and } \frac{d}{c} = \frac{da}{ac}.$$

Then we have $\frac{b}{a} + \frac{d}{c} = \frac{bc}{ac} + \frac{da}{ac} = \frac{(bc + da)}{ac}$, which is the same as what we had previously.

There is a slightly more efficient way to add two rational numbers if we are interested in having the resulting fraction in simplest form. This is a little modification of the last method. What we want is to have a denominator which is the same for both fractions. If we could have a smaller one, we would have less computation. The denominator common to the two must be a multiple of both denominators. So the smallest possible denominator is the least common multiple of the two.

Probably the best way to explain this process is by means of a simple example. Suppose we are faced with the sum $\frac{5}{6} + \frac{7}{15}$. Here the l.c.m. of the denominators is 30 (see Section 4.5). Since 6 must be multiplied by 5 to get 30 we multiply the numerator and denominator of the first fraction by 5 and since 15 must be multiplied by 2 to get 30 we multiply the numerator and denominator of the second fraction by 2. Thus

$$\frac{5}{6} = \frac{25}{30} \text{ and } \frac{7}{15} = \frac{14}{30}$$

and the sum is $\frac{(25 + 14)}{30} = \frac{39}{30}$. Notice that for all our labor the resulting fraction is not in simplest form. However, it is somewhat simpler than the result $\frac{117}{90}$ which we would have obtained using the definition alone. Actually the advantages in using the l.c.m. are much greater in algebraic expressions than for numbers and for this reason there is some point in showing this method to the students. But, in the opinion of the author, it is better to learn the process first without the use of the l.c.m.

Exercises

1. Prove that if 1 is the g.c.f. of b and d, and if the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are in simplest form, then so is the fraction

$$\frac{ad + bc}{bd}.$$

You here may want to refer back to Section 4.2.

2. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions in simplest form. Show that their sum is not an integer unless $b = d$.

3. Assume that all letters below stand for rational numbers and no denominator is zero. Then find which hold for all rational number values of the letters. (It might be helpful first to try out particular numbers):

1) $\dfrac{2}{a + b} = \dfrac{2ab}{\frac{1}{a} + \frac{1}{b}}$

2) $\dfrac{a + b + c}{3} = b + \dfrac{(a - b) + (c - b)}{3}$

3) $\dfrac{2ab}{a + b} = \dfrac{2}{\frac{1}{a} + \frac{1}{b}}$

(Note: the expression on the right side of 3) is called the harmonic mean of a and b. Try it for $a = \frac{1}{2}$, $b = \frac{1}{3}$, for example.)

Problem

1. Prove that $\dfrac{b}{a} + \dfrac{d}{c} = \dfrac{(bc + ad)}{ac}$ when b or d or both are zero. This extends our definition of addition from positive rational numbers to non-negative rational numbers.

5.9 The basic structure of rational numbers

Now that we have completed the basic structure of non-negative rational numbers let us look at what we have. It is understood in the summary below that a, b, c and d are whole numbers and whenever a number appears in a denominator it is not zero.

1. The fraction $\frac{b}{a}$ represents the number with the property:

$$\left(\frac{b}{a}\right)a = a\left(\frac{b}{a}\right) = b \ .$$

2. Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ represent the same number if and only if $ad = bc$.

3. The product of two fractions (or, if you wish to be particular, the fraction which represents the product of the numbers represented by the two fractions) is given by

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd} \ .$$

4. The sum of two fractions is: $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{(ad + bc)}{bd}$.

5. The set of numbers represented by the fractions described above is called the set of non-negative rational numbers.

We chose these definitions so that certain properties (one property in one case and another in another) would hold but at this point we do not know that all the desired properties hold for all the operations described above. To be systematic, before we work with our new numbers too much, we should list all the properties and show them one by one. Here we merely list the properties and verify a few of them. R^+ stands for the set of positive rational numbers and R' the set of non-negative ones. Where R^+ or R' are not indicated it is understood that the letters stand for rational numbers.

a and b in R^+ (or R') implies	For addition	For multiplication
Closure:	$a + b$ in R^+ (or R')	ab in R^+ (or R')
Commutativity:	$a + b = b + a$	$ab = ba$
Associativity:	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Identity element:	$a + 0 = 0 + a = a$	$a \cdot 1 = 1 \cdot a = a$
Inverse element:	Does not exist for addition.	If a is in R^+ there is an element $\frac{1}{a}$ in R^+ such that $a(\frac{1}{a}) = (\frac{1}{a})a = 1$.
Distributive Property:	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$	

Most of these properties follow quite easily from the corresponding properties for the counting numbers, in view of the definitions given. For instance, if $\frac{r}{s}$ and $\frac{t}{u}$ are two rational numbers in which r, s, t, u are counting numbers, their sum is, by definition, $(ru + st)/su$. Since the set of counting numbers is closed under multiplication and addition, $ru + st$ and su are counting numbers and the sum is a rational number. Hence we have shown that the closure property for addition holds.

For the identity elements, note that, using the definition of addition, we have:

$$\frac{0}{1} + \frac{r}{s} = \frac{(0 \cdot s + 1 \cdot r)}{s} = \frac{r}{s} = \frac{r}{s} + \frac{0}{1},$$

using the properties of the counting numbers involving both the additive identity 0 and the multiplicative identity 1. Similarly

$$\frac{1}{1} \cdot \frac{r}{s} = \frac{1 \cdot r}{1 \cdot s} = \frac{r}{s} = \frac{r}{s} \cdot \frac{1}{1}.$$

Probably the hardest is the associative property for addition. We do half of it and leave the other half as an exercise. We want to show

$$\frac{r}{s} + \left(\frac{t}{u} + \frac{x}{y}\right) = \left(\frac{r}{s} + \frac{t}{u}\right) + \frac{x}{y}$$

where all the letters stand for whole numbers and none of the denominators is zero. The left side is equal to:

$$\frac{r}{s} + \frac{(ty + ux)}{uy} = \frac{(ruy + sty + sux)}{suy} .$$

Since the set of positive rational numbers with the operation of multiplication has the properties of closure, associativity, existence of identity and inverse we say that this set forms a multiplicative group. Since, furthermore, multiplication is commutative, it is called a multiplicative abelian (or commutative) group. In fact, any set with an operation (some rule of combination) is said to form a group if it has the four properties named at the beginning of this paragraph. We shall meet other examples of groups later. Notice that the positive rational numbers do not form an additive group since neither the identity element for addition is included nor the inverses. If we were to consider the non-negative rational numbers there would be an identity element but still no additive inverse.

In this connection there is another pair of properties which we should point out and for which we had a precedent in the set of whole numbers:

The well-defined properties

$a = b$ implies $a + c = b + c$ and $ac = bc$.

These seem quite obvious. They are in fact a joint property of the definition of equivalence of fractions and the definitions of addition and multiplication. Our reason for emphasizing them here is that they are basic for the manipulation of equations later on. In the language of Euclid this property of addition was: equals added to equals are equal.

We show this for addition and leave the multiplication as an exercise:

Suppose $\frac{r}{s} = \frac{t}{u}$; we wish to show $\frac{r}{s} + \frac{x}{y} = \frac{t}{u} + \frac{x}{y}$.

Now

$$\frac{r}{s} + \frac{x}{y} = \frac{(ry + sx)}{sy}$$

$$\frac{t}{u} + \frac{x}{y} = \frac{(ty + ux)}{uy} .$$

These two represent the same rational number if and only if

$$(ry + sx)uy = (ty + ux)sy$$

that is

$$ruy^2 + sxuy = tsy^2 + uxsy .$$

But $\frac{r}{s} = \frac{t}{u}$ implies $ru = st$ and thus, by the well-defined property for multiplication of whole numbers $ruy^2 = tsy^2$ and by the well-defined property for the addition of whole numbers (as well as the associative and commutative properties of multiplication):

$$ruy^2 + sxuy = tsy^2 + uxsy.$$

This is what we wanted to prove.

Equally important are the

Cancellation properties

$$\frac{r}{s} + \frac{x}{y} = \frac{t}{u} + \frac{x}{y} \text{ implies } \frac{r}{s} = \frac{t}{u}$$

$$\left(\frac{r}{s}\right)\left(\frac{x}{y}\right) = \left(\frac{t}{u}\right)\left(\frac{x}{y}\right),\ x \neq 0, \text{ implies } \frac{r}{s} = \frac{t}{u}\,.$$

These follow directly from our definitions above and the corresponding properties for the whole numbers. We show how it goes for multiplication.

$$\left(\frac{r}{s}\right)\left(\frac{x}{y}\right) = \frac{rx}{sy}\,;\ \left(\frac{t}{u}\right)\left(\frac{x}{y}\right) = \frac{tx}{uy}\,.$$

Since the two fractions are equivalent we have

$$rxuy = sytx.$$

That is $$ru(xy) = st(xy).$$

Then, $xy \neq 0$ and the cancellation property for the product of whole numbers implies

$$ru = st$$

which, in turn, implies

$$\frac{r}{s} = \frac{t}{u}\,.$$

Finally there is one very important consequence of the closure property of multiplication for the numbers of R^+. This is that if a and b are *positive* rational numbers their product must be a *positive* rational number. Or, in other words, *if the product of two non-negative rational numbers is zero, at least one of them must be zero*.

Problems

1. Prove the associative property for multiplication of non-negative rational numbers.

2. Complete the proof of the associative property for addition of rational numbers.

3. Prove the well-defined property for multiplication of rational numbers.

4. Prove the cancellation property for addition of rational numbers.

5.9 Subtraction of positive rational numbers

Here the situation is quite analogous to that for the counting numbers. Hence it is desirable for the reader to renew his acquaintance with the subtraction of integers. Just as for the whole numbers, the equation $r + x = s$ is sometimes solvable in the set of non-negative rational numbers and sometimes not. There are three possibilities:

1. $r + x = s$ is solvable for a positive rational number x.

In this case we say that s is greater than r and write $s > r$.

2. $r + x = s$ is solvable for $x = 0$, in which case $r = s$.

3. $r + x = s$ is not true for any non-negative rational number x.

Of course then we need to have some way of telling whether or not one rational number is greater than another. For the counting numbers we referred to the number line. For the rational numbers it seems better to deal with them first algebraically for we can thus relate the problem in rational numbers to that for counting numbers.

Suppose $r = \frac{b}{a}$ and $s = \frac{d}{c}$. We can more easily compare them, at least at first, if we write them as fractions with a common denominator. Then our equation $r + x = s$ becomes:

$$\frac{bc}{ac} + x = \frac{da}{ac} .$$

or, if we write $x = \frac{y}{ac}$ we have

$$\frac{(bc + y)}{ac} = \frac{da}{ac} .$$

From problem 1 of Section 5.3, these can be equal only if

$$bc + y = da.$$

In other words we have shown that the following two equations are equivalent if $y = (ac)x$.

(1) $$bc + y = da$$

(2) $$\frac{b}{a} + x = \frac{d}{c} .$$

Notice that if x is a positive number so is y, and if $x = 0$ so is y. Thus we have

1. If equation (2) is solvable for a positive rational number x, that is, if $\frac{d}{c} > \frac{b}{a}$, then $da > bc$. Conversely, if $da > bc$, equation (2) has a positive rational number as its solution and $\frac{d}{c} > \frac{b}{a}$.

2. If equation (2) is solvable for $x = 0$, then $\frac{d}{c} = \frac{b}{a}$ and $da = bc$. The converse holds.

3. If equation (2) is not solvable for a non-negative rational number x, (1) is not solvable for a whole number y. This means that $da < bc$ and the equation $da + z = bc$ is solvable, that is the equation $\frac{d}{c} + x = \frac{b}{a}$ is solvable and $\frac{d}{c} < \frac{b}{a}$.

Briefly, then, we have the following:

1. $\frac{b}{a} < \frac{d}{c}$ if and only if $bc < ad$.
2. $\frac{b}{a} = \frac{d}{c}$ if and only if $bc = ad$.
3. $\frac{b}{a} > \frac{d}{c}$ if and only if $bc > ad$.

Really, of course, the first and third cases are essentially the same.

All this means that if we have any two rational numbers r and s, just as for whole numbers, one is greater than the other or they are the same number. Thus the rational numbers can be put in order on the number line just as the whole numbers. We can compare two numbers $\frac{b}{a}$ and $\frac{d}{c}$ by writing them as fractions with the same denominator and comparing the numerators: or, what is simpler, comparing the products bc and ad.

Just as for whole numbers, if r and s are rational numbers with $r < s$ we write the solution of $r + x = s$ and $x + r = s$ as $s - r$. This tells us

$$\frac{d}{c} - \frac{b}{a} = \frac{(da - bc)}{ac} .$$

5.10 Density

In Section 1.2 we gave a definition of density which, for convenience, we repeat here as:

Definition 1: A set, S, with a relation R having the property of betweenness is called dense if, for any two elements of the set, there is a third element between them.

If S is the set of rational numbers and R is the relation "less than," we can show that the set S is dense by showing that between any two rational numbers s and r (with $s > r$) there is a rational number. In fact, such a number is

$$r + \frac{s - r}{2} = \frac{s + r}{2}.$$

This shows, incidentally, that between any two rational numbers there is an infinite number of rational numbers. Why?

There is another definition of density for a subset of real numbers and the relation of inequality which is more in accord with the usual usage in advanced mathematics. Here is the second one:

Definition 2: Any subset S of the set of real numbers is called dense if between any two real numbers there is a number of S.

Notice that these definitions are not equivalent. In fact, Definition 2 implies Definition 1 since rational numbers are real numbers. There are sets, though rather complex ones, which are dense by Definition 1 but not by Definition 2. To show that the set of rational numbers is dense in this sense we would have to show that between any two real numbers there is a rational number. We can do this a little more easily after we have more experience with inequalities and we postpone this until Section 9.3.

Exercises

1. For each of the following pairs of fractions, find which represents the smaller number:

   a) $\frac{18}{7}$ and $\frac{18}{11}$ d) $\frac{5}{18}$ and $\frac{5}{17}$

   b) $\frac{11}{15}$ and $\frac{7}{15}$ e) $\frac{17}{9}$ and $\frac{15}{7}$

   c) $\frac{17}{9}$ and $\frac{15}{7}$ f) $\frac{17}{9}$ and $\frac{15}{11}$

   Which of the above could be answered by inspection without calculation? Why?

2. Put the following triples of numbers in the proper order from lesser to greater:

   a) $\frac{2}{7}, \frac{11}{17}, \frac{15}{24}$ c) $\frac{2}{7}, \frac{11}{17}, (\frac{2}{7} + \frac{11}{17})(\frac{1}{2})$

   b) $\frac{24}{15}, \frac{17}{11}, \frac{7}{5}$

3. If $a > b$ with a and b whole numbers, which of the following is greater: $\frac{1}{a}, \frac{1}{b}$. Can this be used to simplify some of the work in the two previous exercises?

4. Show that if 1 is the g.c.f. of b and c, there are integers x and y such that

$$\frac{x}{b} - \frac{y}{c} = \frac{1}{bc}$$

(see Section 4.4).

5. One bell rings every third of an hour and another every fifth of an hour. They ring together on the hour. How close together can they ring without ringing simultaneously?

6. Refer to Problem 4 below and write the Farey Series for $n = 10$.

Problems

1. We compared the value of two rational numbers in fractional form by comparing the numerators of two equivalent fractions with equal denominators. Could we have done this by comparing the denominators of two equivalent fractions with equal numerators? If so how?

2. Prove that if $\frac{b}{a}$ and $\frac{c}{d}$ are two positive rational numbers and $\frac{c}{d} > \frac{b}{a}$, then $\frac{(b + c)}{(a + d)}$ is between them.

3. We could have chosen a simpler definition of the sum of two fractions, namely:

$$\frac{r}{s} + \frac{t}{u} = \frac{(r + t)}{(s + u)} .$$

Indeed this is sometimes seen. What would be some peculiar consequences of this definition? Which of the fundamental properties for addition would hold?

4. There is an interesting series called the Farey Series which consists of all the fractions in increasing order between 0 and 1 with denominators less or equal to than a fixed number n. For $n = 7$ the series is

$$0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2},$$

$$\frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, 1.$$

One peculiar property is that if any three successive terms are chosen, the middle one can be obtained from the other two by the process described in Problem 2 above. For instance, if we choose $\frac{1}{5}$, $\frac{1}{4}$, $\frac{2}{7}$ we have

$$\frac{(1 + 2)}{(5 + 7)} = \frac{3}{12} = \frac{1}{4}.$$

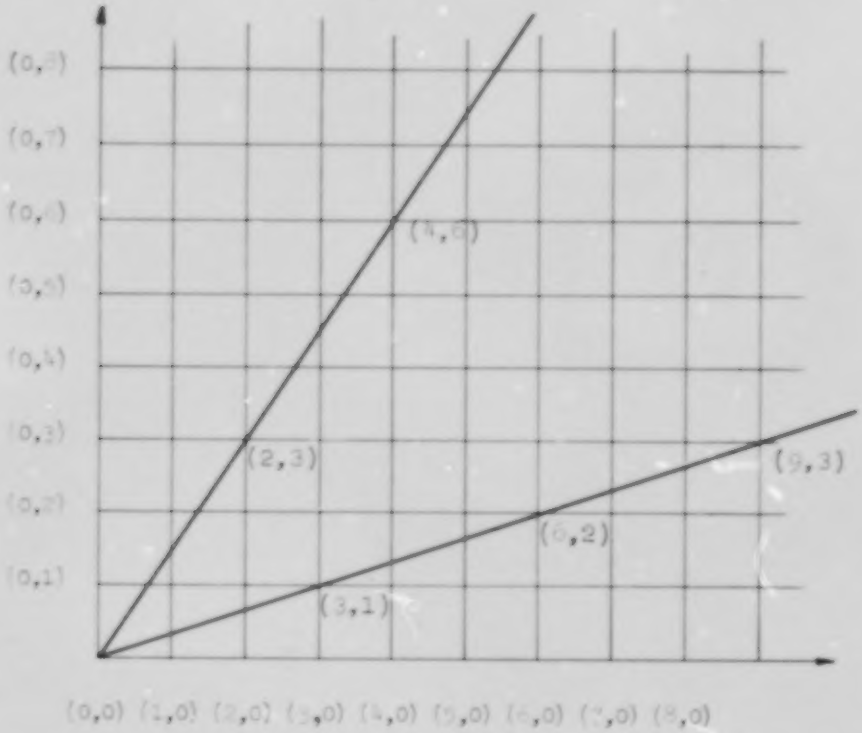
You should check this for some other cases. Another property, and the one which makes it useful in the theory of numbers is that the difference between any two successive numbers of the series is 1 divided by the product of their denominators: e.g., $\frac{3}{7} - \frac{2}{5} = \frac{1}{35}$.

5. If $\frac{a}{b}$ is a fraction in which a and b are counting numbers, will the number which the fraction represents increase, decrease or remain the same if we add the same counting number to the numerator and denominator of the fraction?

5.11 Ordered number pairs

Whether we write a fraction in the form 2/3 or $\frac{2}{3}$ it is an "ordered pair" of numbers (or, if you will, numerals). In the first instance it is ordered from left to right since 2/3 is different from 3/2; in the second instance it is ordered from top to bottom since $\frac{2}{3}$ is different from $\frac{3}{2}$. (Actually in the second case we order the numbers from left to right when we say "two-thirds" instead of "thirds-two.") Another way of writing a fraction could be (2,3) though this seems awkward since we are not used to it. There are two advantages, however, in writing the ordered pair in the form (2,3). The first is that we can ascribe various meanings to it as we please. One time it might mean 2/3, another time 2 - 3, or, again 3 + 2 or 2 · 3. In the latter two cases the ordered pairs would be different but the sum and products would be the same, e.g., the ordered pair (2,3) and (3,2) would be associated with the same number for the sum since 2 + 3 = 3 + 2.

A second advantage of using the more general notation, (a,b), is that we can set up a one-to-one correspondence between these ordered pairs and a set of points in the plane. Consider two number rays perpendicular to each other with the counting numbers marked off on them as indicated. The numerals on the horizontal ray can denote columns and those on the vertical ray rows. We can then form a grid (or lattice) and each point where the lines of the grid cross can be identified by the column and row in which it occurs. Thus the point in the 1 column and the 3 row can be denoted by the number pair (1,3). This will be a different point from that in the 3 column and 1 row which is designated by (3,1). The points on the vertical ray will be in order: (0,0), (0,1), (0,2), (0,3), ... and those on the horizontal ray will be denoted by (0,0), (1,0), (2,0), (3,0),... Every point where the lines of the grid cross, that is, every lattice point will be designated by an ordered pair (x,y) where x and y are whole numbers and every ordered pair (x,y)



of whole numbers will designate a point. The point (0,0) is called the origin. In other words there is a 1 - 1 correspondence between the lattice points and the ordered pairs of whole numbers.

Now return to the interpretation of the ordered number pair (a,b) as the fraction a/b. In this chapter the only fractions we have considered are those in which the numerator is a whole number and the denominator a counting number. So, by the means described in the paragraph above, we have a one-to-one correspondence between all such fractions and the lattice points of the plane, except those on the horizontal ray; these lattice points would correspond to the ordered pairs in which the second element is zero.

This can lead to a correspondence between the rational numbers and lattice points by way of the ordered pairs. Since here the situation is a little more complex, first we find the lattice points which correspond to the integer 3 or 3/1. (The teacher may want to postpone the rest of this section until he considers Chapter IX.) Other fractions representing the same integer are:

$$\frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots, \frac{3n}{n}$$

for any counting number a. If you look at the set of corresponding points of the grid, you will see that they are all on a straight line through the origin and the point (3,1). Similarly, if we consider the pair (2,3) corresponding to the fraction 2/3, then all the pairs which correspond to this rational number will be (2a, 3a) for counting numbers a. If these are drawn on the grid, they will be seen to be the lattice points on the line through the origin and the point (2,3).

In general we shall show in Chapter IX that for any fraction $\frac{a}{b}$, the ordered pairs which are associated with the fractions representing the same rational number will all be on a line through the origin and the point (a,b). Furthermore, the point nearest to the origin will correspond to the fraction in lowest terms, since its coordinates will be the least. Thus we not only have a one-to-one correspondence between the fractions and the lattice points not on the horizontal ray, but to each rational number corresponds a unique non-horizontal line through the origin; conversely to each non-horizontal line through the origin and a lattice point corresponds a unique rational number.

We could even carry the correspondence a little further and "add points". We know, from the sum of two fractions, that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} .$$

If we write this in the form of number pairs we have

$$(a,b) + (c,d) = (ad + bc,bd).$$

So, in this sense the sum of two lattice points is another lattice point. Moreover we can, in a sense, add lines, for suppose the first lattice point is on line L through the origin and the second on line L', then the sum will be on a line L" through the origin. The well-defined property of the sum of two rational numbers tells us that if we replace the first lattice point by another lattice point on L, and the second lattice point by another lattice point on L', our result will be a lattice point on the same line L". For example:

$$\frac{2}{3} + \frac{4}{5} = \frac{10 + 12}{15} = \frac{22}{15} .$$

If we replace $\frac{2}{3}$ by the equivalent fraction $\frac{10}{15}$ and $\frac{4}{5}$ by the equivalent fraction $\frac{28}{35}$ and add the numbers which they represent we get

$$\frac{70 + 84}{105} = \frac{154}{105}$$

which is equivalent to $\frac{22}{15}$. This means that not only do we have a correspondence between rational numbers and lines through the origin but that this correspondence is preserved when we add in the above sense. The same could be seen to be true for multiplication. In such a setting as this, the well-defined property is anything but trivial. It was for this reason that we mentioned it earlier in a simpler setting.

The ordered pair is a way of writing what is called a Cartesian Product. Formally, if W is any set and N another, or the same set, then

$$W \times N$$

stands for the set of ordered pairs (w,n) where w is an element of W and n an element of N. Above, W is the set of whole numbers and N the set of natural counting numbers.

The teacher will have to use his judgement on how far to go with the ordered pair idea at this point, but it is a fundamental point of view and we shall meet it again in various guises.

Problems

1. Why are there no rational numbers corresponding to the points on the horizontal ray described above?

2. Suppose (a,b) were another way of writing a + b. What points of the lattice would represent the number 4, the number 5, in general the number n?

3. Suppose (a,b) were another way of writing ab. What points of the lattice would represent the number [illegible], the number [illegible], the number 7, in general the number n?

4. Suppose (a,b) were another way of writing a - b. What points of the lattice would correspond to whole numbers. What would correspond to the whole number n?

Reference

Parry Series: [illegible] (Section [illegible]), Groups: [illegible] (Chapter [illegible]) and [illegible] (Chapter [illegible]), [illegible] (Chapter 5), [illegible] (Chapter [illegible]), 13 (Chapter [illegible]), [illegible] (Chapters [illegible]), [illegible] (Chapters [illegible]).

Chapter 6

RATIOS, DECIMALS AND APPLICATIONS

6.1 Introduction

Here we are concerned with another way of representing rational numbers which extends to other kinds of numbers as well. Anyone who deals with mathematics knows the usefulness of notation but he also realizes the danger of dissassociating notation from what it really represents. For many teachers this dissassociation has been especially notorious in the case of decimals. But many other teachers have for a long time taught decimals in relationship to the rational numbers and, in fact, the old term "decimal fractions" acknowledged this relationship. It is at this place where the relationships are most important, for a knowledge of decimals and their connection with fractions reinforces knowledge of both.

It is especially important to maintain a proper perspective with relation to applications. There is, first, the choice of applications. These should be largely within the range of the student's actual interest and practice. If he has a bank account, interest is important. If he is concerned with baseball, the percentage of games lost or won is important. But it is a rare junior high school student who pays income tax or who borrows from the bank at a discount.

Second, it is important that each application not be treated as a separate body of knowledge but as having a common base in mathematics. For instance, the two following problems are the same mathematically and the student and teacher should recognize them as such:

Problem 1. If the population of a town is 1000 and it increases in one year by 5%, how many more persons are in the town at the end of the year than at the beginning?

Problem 2. If $1000 is deposited in a savings account and each dollar earns five cents interest over the period of a year, what will be the interest at the end of the year on the total amount?

Finally, it should be kept in mind that it is the fundamental ideas which are important and in no case should applications outdistance their connection with these ideas. The most important thing is that a student have a firm base of knowledge and experience so that he can make his own applications as they arise.

6.3 Ratios

Rational numbers have many kinds of applications. Since the language is different, sometimes we have to think a second time before we realize that it is really a rational number we are dealing with. We hear such things as: one out of five children entering high school graduates from college, the odds are two to three that that horse will place first, the sides of the two similar triangles are in the ratio four to five. Each of the above could be stated in terms of fractions: one fifth of the children entering high school graduate from college, the probability that the horse will place first is two-fifths, each side of one triangle is four-fifths the length of the corresponding side of the other triangle.

A proportion is only a relationship between two fractions. "Two is to five as four is to ten" is merely a different way of saying:

$$\frac{2}{5} = \frac{4}{10}.$$

In more general terms, if a, b and c are proportional to A, B and C we mean that for some fixed number m: $a = mA$, $b = mB$, $c = mC$. This just means that $\frac{a}{A}$, $\frac{b}{B}$, $\frac{c}{C}$ represent the same number, m.

In the old notation: $a:b = c:d$ and the rule that the "product of the means is equal to the product of the extremes" is merely in disguise the fact that $\frac{a}{b} = \frac{c}{d}$ if $ad = bc$; and hence this rule is superfluous.

In all these cases, the important thing is to recognize in the language the meaning of the terminology in terms of the rational numbers. Here it is a matter of translation, and the manipulations should be in terms of fractions.

6.4 Decimals

With decimals there is again a translation problem but, whereas ratios are usually easier to deal with if we consider them as fractions, decimals have their own rules of manipulation which make them useful in themselves. In fact, to add decimals is much easier than to add fractions and it is easier to compare two numbers in decimal form than in fractional form.

The decimal notation, as you know, uses a decimal point as part of the indication of the number represented. Within the United States the decimal point is placed below the line and a raised period denotes multiplication. In many other countries it is the other way around. With a whole number if we use the decimal point at all we place it at the end, that is, the right-hand side of the number. Thus 325 and 325. stand for the same number. Similarly 1 and 1. stand for the same number. Then we indicate division by ten by a change in the position of the decimal point. Thus

$$.1 = \frac{1}{10}, \quad .01 = \frac{1}{100}, \quad .001 = \frac{1}{1000}, \quad .0001 = \frac{1}{10000}.$$

Another way of writing this series would be

$$.1 = 10^{-1}, \ .01 = 10^{-2}, \ .001 = 10^{-3}, \ .0001 = 10^{-4}, \ \ldots,$$

Notice that in all cases the exponent, e.g., -4, corresponds to the number of digits to the right of the decimal point, e.g., 4. This is slightly different from what happens with the positive powers:

$$10 = 10^1, \ 100 = 10^2, \ 1000 = 10^3, \ 10000 = 10^4, \ \ldots,$$

when the exponent counts the number of zeros. We complete the picture by defining:

$$1 = 10^0.$$

Thus:

$$543.1034 = 5 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0 + 1 \cdot 10^{-1} + 0 \cdot 10^{-2} + 3 \cdot 10^{-3} + 4 \cdot 10^{-4}.$$

One advantage of decimals is that not only can every real number be represented as a decimal (see Chapter VIII), but with one type of exception which we shall mention later, the decimal which represents it is unique.

The addition and subtraction of decimals is no problem since the techniques and the reasons are just the same as for whole numbers. But multiplication and division present some additional difficulties.

For instance:

$543200 = (5432)(100)$

$5432. = 5432$

$$543.2 = 543 + \frac{2}{10} = \frac{(5430 + 2)}{10} = \frac{5432}{10}.$$

$$54.32 = 54 + \frac{32}{100} = \frac{(5400 + 32)}{100} = \frac{5432}{100}.$$

$$5.432 = 5 + \frac{432}{1000} = \frac{(5000 + 432)}{1000} = \frac{5432}{1000}.$$

This shows that every time we move the decimal point one place to the left we divide the number by 10 and every time we move it to the right we multiply by ten. Sometimes we have to put zeros in to get the decimal point in the right place just as we had to do for the decimal notation in the beginning: thus $5432 \cdot 10 = 54320$ and $\frac{5432}{10000} = .05432$. So division or multiplication by 10 in the decimal system is easy.

What about multiplication in general? Here again it is better to consider it in terms of an example:

$$543.2 \times 87.03.$$

This is $\frac{5432}{10} \cdot \frac{8703}{10^2} = \frac{(5432)(8703)}{10 \cdot 10^2} = \frac{(5432)(8703)}{10^3}$. For the product, therefore, of 543.2 and 87.03 in each case the exponent of 10 in the denominator above is the number of digits to the right of the decimal point, namely, 1 and 2 respectively. Then the exponent of 10 in the denominator of the product is the sum of these two exponents, in this case 3. So in general we have:

The number of digits to the right of the decimal point in a product of two numbers in decimal form is the sum of the number of digits to the right of the decimal point in the two members of the product.

A somewhat different kind of example is the following product: (5400)(32.123) = 173464.200 = 173464.2. In notation, suppose the number A has a digits to the right of the decimal point and B has b digits to the right, then the product AB has $a + b$ digits to the right of the decimal point.

For division, we just "turn the rule around." Suppose C has c digits to the right of the decimal point and we are to divide by A with a quotient B. Then $a + b = c$, where a and b have the same meanings as in the previous paragraph, is equivalent to

$$b = c - a.$$

That is, we make the number of the digits to the right of the decimal point in the quotient, the difference of the numbers of digits of the other two numerals.

For example, suppose we wish to divide 88.995 by 523.5. To see what is happening, let us write this in terms of fractions:

$$88.995 \div 523.5 = \frac{88995}{1000} \div \frac{5235}{10} = \frac{88995}{1000} \cdot \frac{10}{5235}$$
$$= \frac{88995}{5235} \cdot \frac{1}{100}.$$

So, for our result, we first divide 88995 by 5235 and then divide the result by 100. This is represented by moving the decimal point in the quotient

$$\frac{88995}{5235} = 17$$

two places to the left, giving us 0.17.

In practice, this is of course, shortened. Since 523.5x = 88.995, x being the quotient, the number of digits to the right of the decimal point in the product (in this case 3) is 1 more than that for the quotient x.

Hence the number of digits to the right of the decimal point in the quotient is 3 - 1 = 2. Mechanically, this is accomplished in various ways.

6.4 Scientific notation and approximation

Closely connected with the above is a notation, frequently used in science, which has two purposes: one the concise representation of very large or small numbers and also the indication of the degree of accuracy in a number representing a measurement. For both, powers of ten are used, as follows:

$$53{,}000{,}000{,}000 = 53\cdot 10^{9} \quad , \quad .0000000000053 = 53\cdot 10^{-13}.$$

As regards accuracy we must remember that there are two kinds of uses of numbers in mathematics. When we deal with numbers in the abstract and sometimes in the specific there are no approximations involved. When there are ten people in a room we can be sure that there are not 10.1 people or 9.99. There are just exactly ten people. But for the population of a city we cannot be sure. We know that the number must be a whole number but which whole number it is we can only approximate. Also when we measure the length of a table we can be pretty sure that the measurement is accurate to within an inch or even an eighth of an inch but a more accurate measurement would require a finer instrument of measurement, and any instrument has its limitations in accuracy. So, when a measurement is given, it is important to know how accurate it is.

Now if a measurement is given as 53000 feet it is not clear whether it is accurate to within a thousand feet, a hundred feet, ten feet or a hundredth of a foot. In the respective cases these can be represented as

$$53\cdot 10^{3},\ 530\cdot 10^{2},\ 5300\cdot 10,\ 5300000\cdot 10^{-2},$$

indicating the accuracy by the multiple of the power of ten.

This can also be used to approximate the accuracy of a product. Here one can consider absolute error or relative error. We illustrate them each by examples.

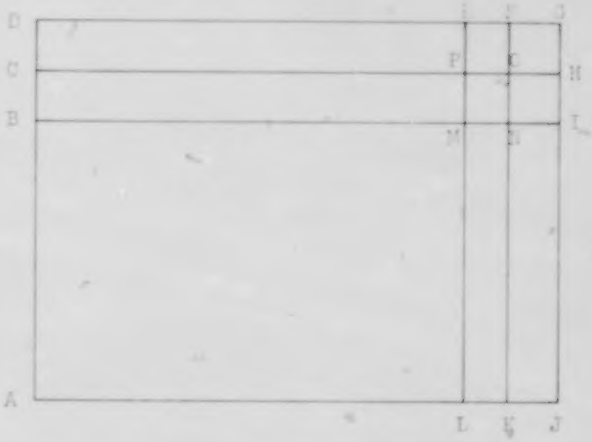
Suppose 523.4 and 78.5 represent two measurements in feet. They are presumably accurate to within one-tenth of a foot. How accurate is their product in square feet? Presumably the first is between 523.35 and 523.45 and second between 78.45 and 78.55. So the product will be between 523.35 × 78.45 and 523.45 × 78.55 that is between

$$41056.8075 \text{ and } 41116.9975.$$

This means that in this case the product is accurate to within one half of the difference between these two numbers, or about 30.1 square feet. In general, designate the measurements by a and b. Then if we assume the maximum possible error is the same in both cases, we see that the true value of the area will have to lie between the product of the least possible dimensions: a - e and b - e, and the product of the greatest possible dimensions, a + e and b + e. That is, the area must lie between

$$(a-e)(b-e) = ab - e(a+b) + e^2 \text{ and } (a+e)(b+e) = ab + e(a+b) + e^2.$$

This can be seen also from the adjoining figure if we take a to be the



length of the line segment $\overline{AB}$ and b that of the segment $\overline{AK}$, with e the length of the segments: $\overline{BC}$, $\overline{CD}$, $\overline{LK}$, $\overline{KJ}$. Then e^2 will be the area of the square MPON which will be much smaller than the area of the rectangle LPOK (which is ea) as well as the area of the rectangle CBNO (which is eb) provided that a and b are much larger than e. So, if we disregard e^2 we see that the maximum error is about $e \cdot (a+b)$, which, in the case of the example above is .05(523.4 + 78.5), that is, approximately 30.

From another point of view, it is the relative error (or ratio of error) which is more pertinent. It would be much harder to achieve accuracy to one foot in measuring 5134 feet than in measuring 51.34 feet. So in practice we might describe the possible error in terms of a ratio: for example, one part in a hundred. Thus if e is now the ratio of possible error, ea would be the maximum error, that is, the actual measurement of the side $\overline{AC}$ would be between $a - ea$, the length of segment $\overline{AB}$, and $a + ea$, the length of segment $\overline{AD}$. We could set up a similar correspondence for b on the assumption that the possible relative error is the same. Hence the area would lie between

$$(a - ea)(b - eb) = ab - 2eab + e^2ab$$

and

$$(a + ea)(b + eb) = ab + 2eab + e^2ab.$$

This time e^2ab will be the area of the little rectangles MPON and GHOF. So the maximum possible error will be about $2eab$ and the relative possible error

$$\frac{2eab}{ab} = 2e.$$

Thus the relative possible error of the product is twice that of each measurement.

Two comments should be made in connection with the above discussion. In each case we neglected a term involving e^2 with the remark that it is smaller than e. In fact, the smaller the error, the greater the disparity between e and e^2 as the following table shows:

e	.1	.01	.001
e^2	.01	.0001	.000001

Second, in multiplying numbers which represent approximations there is no point in having any regard for the last decimal places. For instance the product of the numbers

$$523.4 \text{ and } 78.5 \text{ is } 41086.90.$$

But if the two numbers represent two measurements in feet which can be in error by as much as .05 of a foot, the area can be in error by as much as 30 square feet. So if the product is to indicate the degree of accuracy, we should write it as

$$411 \cdot 10^2,$$

which indicates that the area is 41,100 square feet to within 30 square feet.

So one has a choice of multiplying the two numbers and then writing the result, taking into account the possible error, or he may take this into account in the multiplication process and abbreviate his multiplication as is illustrated by the following:

$$\begin{array}{r} 523.4 \\ 78.5 \\ \hline 260. \\ 4184. \\ 36638. \\ \hline 41082. \end{array}$$

For this, since we know the answer can be off as much as 30 square feet we certainly do not need to have regard for anything less than 1 foot. So for the first multiplication we multiply .5 by 520. Then we multiply 8. by 523. and 70 by 523.4. After we get our result, we write it to the nearest hundred feet and have $411 \cdot 10^2$.

Exercises

1. How should a measurement of 32 feet be written in scientific notation if it is accurate to the extent indicated below:
   a) To within .1 of a foot.
   b) To within ten feet.
   c) To within .0001 feet.
2. In each of the cases in Exercise 1, what is the approximate relative percent of error?
3. For each of the cases in Exercise 1, what is the greatest possible error?
4. Show why the greatest possible error in a sum is the sum of the greatest possible errors in the members of the sum.
5. If the percentage of error in the measurement of the side of a cube is 1%, what is the approximate possible percentage of error in the computation of the volume of the cube?

6.5 Decimal expansions

We know that $\frac{2}{5} = .4$ and $\frac{1}{8} = .125$ but $\frac{1}{3} = .33333 \ldots$. In other words, there is no finite decimal which represents $\frac{1}{3}$ or is the decimal expansion of $\frac{1}{3}$. We can approximate $\frac{1}{3}$ as closely as we please by a decimal since

$$\frac{1}{3} - .3 = \frac{1}{30}$$

$$\frac{1}{3} - .33 = \frac{1}{300}$$

$$\frac{1}{3} - .333 = \frac{1}{3000}$$

and so on.

In fact:

$$\frac{1}{3} - .333\ldots3 = \frac{1}{3\cdot 10^n}$$

where n is the number of 3's which appear in the decimal. This means that by taking n large enough, that is, using a sufficiently large number of 3's, we can make the difference between $\frac{1}{3}$ and the finite decimal as small as we please. We say that the decimal expansion converges to $\frac{1}{3}$.

Some other decimal expansions of rational numbers are:

$$\frac{1}{11} = .090909\ldots, \quad \frac{1}{37} = .027027\ldots,$$

$$\frac{1}{7} = .142857142857\ldots\ .$$

In all cases there is a succession of digits which repeats: in the first case 09, in the second 027 and in the third 142857. We call such decimals repeating or periodic decimals and the repeating part the repetend.

Of course, in some cases the decimal does not repeat from the beginning. For instance:

$$\frac{4119}{9990} = \frac{1373}{3330} = .4123123123\ldots\ .$$

Every rational number has a repeating or terminating decimal. Why?

Conversely, every repeating or terminating decimal represents a rational number. If the decimal terminates the result is easy. If not, we show what happens in general by an example.

To find the number represented by 5.234234..., let x = 5.234234...

$$\begin{aligned} 1000x &= 5234.234234\ldots \\ \underline{-\quad x} &\underline{{}= \quad\ 5.234234\ldots} \\ 999x &= 5229. \end{aligned}$$

$$x = \frac{5229}{999} = \frac{581}{111}\ .$$

Here is another case in which a repeating decimal does not repeat from the beginning but will after a certain point.

Repeating decimals have many interesting properties which we do not have space to consider here, but two should be mentioned. Since every rational number has a repeating decimal expansion, one which does not repeat without end does not represent a rational number. For instance:

$$.101001000100001\ldots$$

does not represent a rational number.

Also it is possible for two decimals, one of them infinite, to represent the same number. For instance: .999... converges to the number 1. It can be seen from this that any finite decimal represents the same number as an infinite decimal obtained from the finite decimal by decreasing the last digit by 1 and adjoining an infinite succession of 9's.

Exercises

1. Find each of the following products:

   a. $(23 \cdot 10^{7})(32 \cdot 10^{6})$

   b. $(16 \cdot 10^{-4})(15 \cdot 10^{-3})$

   c. $(125 \cdot 10^{8})(8 \cdot 10^{-6})$

   d. $\dfrac{125 \cdot 10^{8}}{5 \cdot 10^{4}}$

   e. $\dfrac{125 \cdot 10^{-3}}{5 \cdot 10^{-5}}$

   f. $\dfrac{34 \cdot 10^{6}}{17 \cdot 10^{-3}}$

2. The dimensions of a rectangle are 6.5 feet and 7.8 feet as measured. If it is known that these measurements are accurate to 1%, what can be the approximate percentage of error in the computed area of the rectangle?

3. Suppose each of the measurements in the previous exercise is accurate to .01 of a foot, what would be the maximum possible error in the computed area?

4. Find the decimal expansion of $\frac{1}{13}$ and show that the decimal repeats. What happens if one multiplies the repeating part of the decimal by the numbers from 1 to 12 inclusive, in comparison with what happened for $\frac{1}{7}$ in Problem 2?

5. Find the decimal expansion of $\frac{1}{17}$ and answer the questions in the previous exercise.

6. Find the decimal expansion of $\frac{1}{19}$.

Problems

1. Show that every rational number has either a terminating decimal expansion or an infinite decimal expansion which from some point on repeats without end.

2. If 142857 (the repeating part of the expansion of $\frac{1}{7}$) is multiplied by 2 we get 285714, by 3 we get 428571. We have the same succession of digits cyclically permuted. Why?

3. Is there any connection between the test for divisibility by 9 and the expansion of $\frac{1}{9}$? Similarly for 11. What would be a test for divisibility by 37?

4. What rational numbers have finite decimal expansions?

5. What would be the expansion of one-seventh in the numeral system to the base seven? What would be the expansion of one-fifth? Does either repeat; does either terminate? In this numeral system, moving the point one place to the left does what to the number?

6. Find two decimal expansions for $\frac{1}{5}$. Does $\frac{1}{3}$ have two decimal expansions?

7. As above, let $x = 1 + 2 + 2^2 + 2^3 + \ldots$ and see that $2x = 2 + 2^2 + 2^3 + 2^4 + \ldots$. Then $2x - x = -1$, or $x = -1$. What is wrong?

6.6 Percentage

Percentage is just another notation for a fraction, but it is in very common use. You know that 5% means $\frac{5}{100}$ or .05 and with this knowledge if you know decimals and rational numbers you know percents. Applications are many and the reader is referred to other books for these except for the following development of compound interest.

If $100 is put into a savings account, it accumulates interest over certain specified periods. For instance, if the rate is stated as 4% per year compounded quarterly, this means that the rate is 1% per quarter. Then, for compound interest, at the end of each quarter 1% of the amount at the beginning of the quarter is called the interest and added to the amount. The amount at the beginning of a period is called the principal and that at the end the amount. Thus we have the following table

Quarter	1	2	3	4	5
Principal at beginning	$100	101	102.01	103.03	104.06
Interest	1.	1.01	1.02	1.03	1.04
Amount at the end	$101	102.01	103.03	104.06	105.10

The simplest way to compute the amounts is by use of tables, but there is a formula. To obtain it we need merely notice that each amount is 1.01 times the previous one. Thus the amounts at the ends of the quarters above are:

$$100(1.01),\ 100(1.01)(1.01) = 100(1.01)^2,\ 100(1.01)^2(1.01)$$
$$= 100(1.01)^3,\ 100(1.01)^3(1.01) = 100(1.01)^4$$

In general, if P is the principal ($100) above, r is the rate per interest period (.01 above) and n the number of periods, the amount at the end of each period is (1 + r) times the amount at the beginning of the period. Thus the amount at the end of the first period will be

$$P(1 + r),$$

at the end of the second period will be

$$P(1 + r)(1 + r) = P(1 + r)^2,$$

and at the end of the third period will be

$$P(1 + r)^2(1 + r) = P(1 + r)^3.$$

Thus the number of times (1 + r) appears as a product will be equal to the number of interest periods. Hence if the rate per interest period is r and the number of interest periods is n, we have as the final amount:

$$P(1 + r)^n.$$

There is another useful way to write this same compound interest formula. Suppose the annual rate is t and the number of interest periods per year is q. Then the rate per interest period will be $\frac{t}{q}$ and the number of interest periods in n years will be qn. For these meanings of the letters the formula for the amount becomes

$$P(1 + \frac{t}{q})^{qn}.$$

For instance, if the annual rate is 4% and interest is compounded quarterly, that is there are four interest periods per year, the amount at the end of five years would be

$$P(1 + .01)^{20}.$$

Fortunately there are tables for these values.

Now let us see what happens in a particular case as q, the number of interest periods, increases with the annual rate remaining the same. Let t, the annual rate, be 6% or .06 and let q be the number of interest periods per year. Thus if the interest is compounded annually, q = 1: while if it is compounded quarterly, q = 4, etc. In the table, A denotes the amount at the end of one year for a sum of $100 deposited at the beginning of the year. So the first line gives values of q, the second line the corresponding values of $\frac{A}{100}$ according to the formula

$$\frac{A}{100} = (1 + \frac{.06}{q})^q$$

and the third line gives A to the nearest cent.

q	1	2	4	6	12	24
$\frac{A}{100}$	(1.06)	$(1.03)^2$	$(1.015)^4$	$(1.01)^6$	$(1.005)^{12}$	$(1.0025)^{24}$
A	\$106.00	\$106.09	\$106.14	\$106.15	\$106.17	\$106.18

Notice that there is only 18 cents difference between the first and last amount and as the number of periods increases the difference in amounts becomes less. As a matter of fact in this particular case, the amount would be \$106.18 for any greater number of interest periods, though if the principal had been \$1000 there would be a little difference. This then naturally leads one to wonder what would happen if the number of interest periods were to increase without limit. The formula for the limiting amount turns out to be

$$Pe^{nt}$$

where e is an important mathematical constant whose value is approximately 2.718, n is the number of years and t is the rate per year. It may be seen from a table that

$$e^{.06} = 1.0618$$

correct to four decimal places. This verifies the statement above that if the number of interest periods per year is more than twelve the amount at the end of the year for \$100 would be \$106.18.

This is a formula for what is called "continuous interest" and is being used in an increasing number of banks today. It has a number of advantages: the formula holds when n is not a whole number and by reference to tables one can find the amount at each instant, it has "good customer appeal" since money draws interest for the exact time it is in the bank, and the slightly increased cost to the bank in interest is more than made up for by the convenience to its accounting department.

Exercises

1. An insurance agent makes a commission of 3% on all insurance sold. How much insurance must he sell to have an income of \$10,000?

2. A town has a population of 1000 persons. Each year for a period of five years, the population increases by 60 persons. What is the percentage of increase each year?

3. Suppose in Exercise 2, the population decreased by 40 persons each year. What would be the percentage of decrease each year?

4. Comparing the two exercises, is the percentage of decrease in any year for Exercise 3 equal to the percentage of increase for Exercise 2? Give reasons why you should have expected the answer to be what it is before you carried out the computation.

5. Suppose in Exercises 2 and 3 we compute the percentage of increase and decrease comparing the original population of 1000 with that at the end of five years? Are the two percentages the same? If so, how does that jibe with the results of Exercise 4?

6. State an exercise involving the same calculations as Exercise 1, but starting with "A man has $1000".

7. In a certain country, half the population is under 16 years of age at present. During the next sixteen years each of these will have an average of two children (that is, each couple will have four children). If the net increase of the rest of the population is 20% over the next sixteen years, what will be the increase in total population at the end of that time?

Problems

1. The population of a certain city increased by 6% in one year and then decreased by 6% in the following year. Does this mean that over the two year period its population neither increased nor decreased?
2. In city A the population increased by 12% over a two year period. In city B the population increased by 6% each year for two years. If the two cities had equal populations at the beginning of the two year period, how did their populations compare at the end of the two years?

3. A person desires to borrow $400 from a bank and will pay it back in four quarterly installments over the period of a year. Since the interest rate for the bank is 6%, the year's interest on $400 is $24 and hence the bank requires that principal and interest be paid off in quarterly installments of $106 each (one-fourth of 400 + 24). Is the person really paying interest at the rate of 6% annually?

References

1, 8 (Chapter 4: Sections 4 and 5), 9 (Chapter 8), 22 (Chapter 9).

Chapter VII
RATIONAL NUMBERS

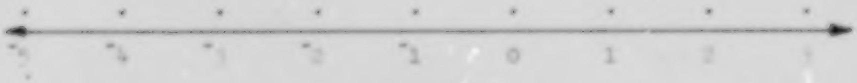
7.1 Introduction

In Chapter V we extended our number system so that we could solve the equation $ax = b$ where a and b are whole numbers and $a \neq 0$. The resulting system was the set of non-negative rational numbers. And we found that the equation $ax = b$ was solvable also in this system provided, again, that a is different from zero. When we consider the equation $a + x = b$ with $b < a$, we come to the negative integers and then the negative rational numbers. This extension is somewhat more difficult because the need of it is not quite as apparent. In the author's opinion part of the reason for the difficulty is that many times a teacher is so anxious to make the topic clear that he (or she) buries the student under a wealth of applications and interpretations. It would be much better to show the algebraic reason and the geometrical interpretation in whichever order seems best. Then the teacher could connect it with one really simple and direct application like the usual thermometer which is the number line placed vertically. Other applications can be confusing except those which the students may suggest themselves and should be left until after the ideas are fixed.

Fundamentally the point of view of the extension is the same as for the rational numbers: the whole numbers are not adequate for our needs; so we introduce new numbers for the purpose and so define equality and the operations that as many of their properties as possible are preserved, and at the same time make these new numbers applicable to the model we have in mind. Here the reader may want to refresh his mind on what went on in Chapter V.

7.2 Definition of negative integers

Recall that in Chapter III, we considered the equation $a + x = b$ where a and b are whole numbers. We found that it had a solution which was a whole number whenever $b \geq a$, and we called that solution $b - a$. In our approach to rational numbers we first considered the more limited equation $ax = 1$ in place of $ax = b$. So here we consider the equation $a + x = 0$, the additive identity. Just as we defined numbers $\frac{1}{a}$ for the first equation, we define new numbers ${}^{-}a$ for the second. Thus for every whole number a, we create a new number ${}^{-}a$ which has the property that ${}^{-}a + a = 0$. We can represent these numbers and the old ones on a number line (instead of ray) as follows:



Thus the point corresponding to ${}^{-}5$ (which we can call the point ${}^{-}5$) is that which is 5 spaces to the left of the point 0. The number ${}^{-}5$ has the property that ${}^{-}5 + 5 = 0$ and the point 0 is reached when we start at the point ${}^{-}5$ and count five spaces to the right. We want addition to be commutative and so we agree that $5 + {}^{-}5$ shall also be zero as well as $a + {}^{-}a = 0$. What about ${}^{-}0$? We know that $0 + 0 = 0$ and thus we should choose ${}^{-}0$ to be the same as 0. Thus for every whole number a, we have defined an opposite number ${}^{-}a$ so that

$$a + {}^{-}a = {}^{-}a + a = 0,\ {}^{-}0 = 0.$$

These new numbers:

$${}^{-}1,\ {}^{-}2,\ {}^{-}3,\ {}^{-}4,\ \ldots$$

we call the negative integers. These together with the whole numbers form the set of integers. The natural numbers are called positive integers. Here the mathematical and practical needs go side by side. We speak of "minus five" in temperature when it has to rise by five degrees to be at zero.

We have used the word "opposite" which carries a connotation of symmetry. If A is opposite B, then B should be opposite A. Thus it is time to define "opposite number" in symmetrical form.

Definition: Two numbers are called opposites if their sum is zero. If n is an integer we denote its opposite by ${}^{-}n$.

We have already defined ${}^{-}p$ in accordance with this definition if p is a whole number. Suppose $p = {}^{-}c$ where c is a positive integer. Then the opposite integer to ${}^{-}c$ would have to be c since ${}^{-}c + c = 0$. Thus

$${}^{-}({}^{-}5) = 5 \text{ and } {}^{-}({}^{-}a) = a.$$

We have defined negative integers. Next we consider addition. Here it is best to take little steps and then summarize our results.

First, suppose a and b are natural numbers with $b \geq a$; what is ${}^{-}a + b$, or, rather, what do we want it to be? Since we want the associative property, we should have

$$a + ({}^{-}a + b) = (a + {}^{-}a) + b = 0 + b = b.$$

But we know that $b - a$ is the number which has the property that if it is added to a the result is b. So we have

$$^-a + b = b - a \text{ if } b \geq a.$$

We want the commutative property and so we agree that

$$b + {}^-a = b - a \text{ also, if } b \geq a.$$

Second, what should be the number $^-a + b$ if $b < a$? Of course the number is $^-a + b$ but we want to express it as a single numeral. Here there are at least two possible approaches and preferences may vary. We first consider the algebraic approach.

$$^-a + b + {}^-b + a = {}^-a + (b + {}^-b) + a = {}^-a + 0 + a = {}^-a + a = 0$$

if the associative and commutative properties are to hold. This means that $^-a + b$ and $^-b + a$ are opposite numbers. We already know that $^-b + a$ is $a - b$ since $a > b$. Hence

$$^-a + b = {}^-({}^-b + a) = {}^-(a - b)$$

To summarize these two cases, we have for a and b whole numbers:

$$b + {}^-a = {}^-a + b = b - a \text{ if } b \geq a.$$

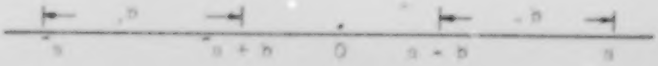
$$b + {}^-a = {}^-a + b = {}^-(a - b) \text{ if } b < a.$$

In fact, for uniformity's sake we define

$$b - a = b + {}^-a$$

regardless of the relative size of b and a but the interpretation is that above. This is natural because $b + {}^-a$ has the property that when a is added to it we get b and $b - a$ has the same property.

We may also use the number line to find a meaning for $^-a + b$ when b and a are whole numbers. To do this recall that to get $a + b$ from a we add b, or on the number line count b spaces to the right. So to add b to ^-a we should start at ^-a and count b spaces to the right. Then our resulting number is $b - a$ if $b \geq a$, as it should be. Now also if a is greater than b, to find the sum of $^-a + b$ we should start at ^-a and count b spaces to the right. In that case we stop short of 0 just $a - b$ spaces, which corresponds to the number $^-(a - b)$. The case in which a is greater than b is illustrated in the figure below:



Here b is the distance between the points ^-a and $^-a + b$ as well as that between the points a and a - b. It is apparent from this that a - b and $^-a + b$ are opposite points.

Thus, whether we consider the addition of a positive and a negative integer algebraically or geometrically we come to the same conclusions.

For addition it remains to consider the sum of two negative integers. Here, in the author's opinion, the algebraic approach is better. Again choose a and b whole numbers and see

$$^-a + {^-b} + a + b = (^-a + a) + (^-b + b) = 0 + 0 = 0$$

if the associative and commutative properties are to hold. This means

$$(^-a + {^-b}) + (a + b) = 0$$

and hence

$$^-a + {^-b} = {^-(a + b)}.$$

On the number line this means that to get the sum $^-a + {^-b}$ you start at ^-a and move b spaces to the left.

Here rules for addition of integers are much more complex than the actual use of them. To summarize them we write first a numerical example and follow it with a literal one. In the latter, for the time being, we let a, b and c designate whole numbers:

i) $^-3 + 5 = 5 + {^-3} = 5 - 3 = 2$ $\qquad$ $^-a + b = b + {^-a} = b - a$ if $b \geq a$.

ii) $^-5 + 3 = 3 + {^-5} = 3 - 5 = {^-2}$ $\qquad$ $^-a + b = b + {^-a} = {^-(a - b)}$ if $b \leq a$.

iii) $^-5 + {^-3} = {^-3} + {^-5} = {^-8}$ $\qquad$ $^-a + {^-b} = {^-b} + {^-a} = {^-(a + b)}$.

We shall see below that i) and ii) hold without the restrictions given.

Here we must distinguish between the process by which we come to these conclusions and the conclusions themselves. We have shown that to be consistent with previous properties, we must define the sum of two integers as we have done. When it comes to actual manipulations with integers we memorize certain rules to the point where their use is second nature and we wonder why anyone should have difficulty with them since we know them so well. An interesting approach to a class would be to introduce the negative integers as opposites of the positive ones and then ask the class what it thinks $^-3 + 5$, $^-5 + 3$ and $^-5 + {^-3}$ should be. It would not be hard to eliminate all but the right conclusions. Here in this section and in the one to come, the author makes no recommendation that the methods used here be used in a junior high school class but rather that these properties be in the mind of the teacher as fundamentally the reasons for the properties developed.

7.3 Subtraction of Integers

We set out to create the negative integers so that we could solve the equation:

$$a + x = b.$$

Have we accomplished this? We certainly have if a and b are whole numbers for the solution is $x = {}^{-}a + b$. This has meaning since ${}^{-}a$ was defined for a a whole number and if a is a negative number, ${}^{-}a$ is its opposite which is a positive number; similarly we defined above the sum of any two integers. Assuming the associative property for integers:

$$a + ({}^{-}a + b) = (a + {}^{-}a) + b = 0 + b = b.$$

This shows that ${}^{-}a + b$ is a solution of $a + x = b$, no matter what integers a and b are.

We now proceed to eliminate the conditions imposed for i), ii), iii) above. So far notice that we have defined $r - s$ only when r and s are whole numbers. Now we know that $s + x = r$ is always solvable in integers. Hence we define $r - s$ to be that solution. So by this definition:

$${}^{-}a + b = b - a$$

for all integers a and b. Also:

$${}^{-}(a - b) + (a - b) = 0$$

by definition,

$$(b - a) + (a - b) = b + {}^{-}a + a + {}^{-}b = 0 + 0 = 0.$$

Thus

$$b - a = {}^{-}(a - b)$$

for all integers a and b. Thus we have for all integers a and b:

$${}^{-}a + b = b + {}^{-}a = b - a = {}^{-}(a - b).$$

As for (iii), the same argument we used when a and b are whole numbers applies for all integers a and b. Thus we have the same conclusion. We summarize the results of this and the previous section, where it is understood that a and b are integers:

i) The solution of $a + x = b$ is $b - a$. This can also be written:

$$b + {}^{-}a,\ {}^{-}a + b,\ {}^{-}(a - b),\ {}^{-}(a + {}^{-}b)$$

ii) ${}^{-}({}^{-}a) = a$.

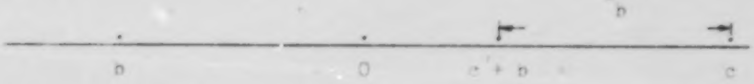
This means that in most cases we can dispense with the elevated minus sign. But there still is a residue of the old meaning left, for while $b - a$ can mean equally "add minus a to b" or "subtract a from b", yet $-a$ by itself can only mean "a number which is the opposite of a".

To illustrate the above consider $7 - (^-3)$. This is equal to $7 + {}^-(^-3) = 7 + 3 = 10$. Also $^-5 - (^-7) = {}^-5 + {}^-(^-7) = {}^-5 + 7 = 7 - 5 = 2$.

It is helpful to see what all this means on the number line. First suppose b is a whole number and c is any integer. This was dealt with in the previous section where we saw that to get the point corresponding to $c + b$ we find the point corresponding to c and count b spaces to the right. Now suppose b is a negative integer. Then

$$c + b = c - (^-b).$$

This will be the number with the property that if you add the positive integer ^-b to it you get c (see figure). Thus the point corresponding to this number must be ^-b points the left of the point which corresponds to c. Thus adding b, where b is negative, is equivalent to moving ^-b points to the left. To summarize the geometrical significance we have:



> To move from the point corresponding to c on the number line to that corresponding to $c + b$, move b spaces to the right if b is positive or ^-b spaces to the left if b is negative.

This is independent of whether c is positive or negative or zero.

7.4 Multiplication of negative integers

Actually multiplication is a little easier than addition. What should $3(^-5)$ be? It should be

$$^-5 + {}^-5 + {}^-5 = {}^-15.$$

Similarly if a and b are whole numbers $a(^-b)$ should be $^-(ab)$. We define it that way. We want multiplication to be commutative and so we define

$$(^-b)a = a(^-b) = {}^-(ab).$$

We could also arrive at this result algebraically by noticing that

$$3(^-5) + 3(5) = 3(^-5 + 5) = 3 \cdot 0 = 0.$$

Thus when we add $3(^-5)$ to 15 we get zero, which shows that $3(^-5) = {}^-15$.

The value for $(^-3)(^-5)$ is best determined by our desire to have the distributive property hold.

$$(^-3)(^-5) + (^-3)(5) = (^-3)(^-5 + 5) = (^-3) \cdot 0 = 0.$$

But $(^-3)(5) = {}^-(15)$ and since $15 + {}^-(15) = 0$, we must have

$$15 = (^-3)(^-5).$$

For the product of two negative integers, the geometrical picture is not much help. In summary we have the following for a and b whole numbers:

$$(^-a)b = a(^-a) = {}^-(ab)$$

$$(^-a)(^-b) = ab.$$

In words we have: the product of two negative integers or the product of two positive integers is positive and the product of a negative integer and a positive one is negative. It is in "explaining" that the product of two negative integers is a positive integer that many ingenious devices are used, like walking backwards on a moving train. This is one place where, in the author's opinion, illustrations of this kind confuse rather than illuminate. The question: "what is the product of two negative integers?" is a mathematical question and deserves a mathematical answer.

It is not very difficult to show that the distributive property holds for the set of integers but if we were to consider all the possibilities the proof would be rather long and somewhat dull. So we merely assume it here.

Exercises

1. For what natural numbers n is $(-1)^n = 1$ and for what values of n is $(-1)^n = -1$?
2. Will the product of 25 negative numbers be positive or negative? or may it be sometimes one and sometimes the other?
3. Will the sum of 25 negative numbers be positive or negative or may it be sometimes one and sometimes the other?
4. Answer the questions of Exercises 2 and 3 if 25 is replaced by 30.
5. The product of 13 numbers is negative. Which of the following statements cannot be true:

   a. all are positive

   b. exactly one is negative

   c. exactly two are negative

   d. exactly three are negative.

   What are the possibilities?
6. Suppose $\frac{1}{5}$ is written 5^{-1} and similarly $\frac{1}{a} = a^{-1}$ for every rational number $a \neq 0$. What would be equal to $(5^{-1})^{-1}$?

7. Using the notation in Exercise 6, that is, letting $\frac{1}{a} = a^{-1}$ whenever $a \neq 0$, group the following into sets of equal numbers:

$$(-5)^{-1},\ -(5^{-1}),\ ({}^{-}5),\ (-\tfrac{1}{5}),\ 5,\ (\tfrac{1}{5})^{-1},\ \frac{1}{(-5)} .$$

8. Let Rx mean "replace x by its reciprocal" and Sx mean "replace x by its opposite", that is, "change the sign." If x = 5 find the following:

$$R5,\ S5,\ R(S5),\ S(R5),\ R(R5),\ S(S5),$$
$$S(R(S5))).$$

Show that each of those in the previous list is equal to one of the following four: 5, R5, S5, R(S5)

Problems

1. Prove $({}^{-}a)(b) = {}^{-}(ab)$ when a is a positive integer and b a negative one, also when a is a negative integer and b a positive one; finally when both a and b are negative.

2. The previous problem for the product $({}^{-}a)({}^{-}b) = ab$.

3. Prove $(a - b) - c \neq a - (b - c)$, but

$$(a - b) - c = a - (b + c) = (a - c) - b$$

7.5 Absolute value

There is one concept which is convenient on several occasions and is very simply defined, namely the absolute value. The absolute value of a number b is written $|b|$ and defined like this:

If b is positive, $|b| = b$.

If $b = 0$, $|b| = 0 = b$.

If b is negative, $|b| = {}^{-}b$.

It has a definite geometrical meaning if the number is thought of as represented on the number line. The absolute value of b is the "distance" of the point which represents b from the point 0. Of course "distance" means the number of units. An important property of the absolute value is

$$|ab| = |a| \cdot |b|.$$

This is easily verified.

In terms of this notation it is easy to show another property of the integers. Let I* denote the set of integers with zero omitted. This set is closed under multiplication for if a and b are any two integers in I*, $|ab| = |a| \cdot |b|$. The right side of this equation is the product of two counting numbers and we know that this product is a counting number, not zero. Then since the absolute value of ab is not zero, the value of ab is not zero either.

Exercises

In this set of exercises, the small letters
stand for integers.

1. Prove that if b and c are both negative or both positive, $bc = |bc|$.
2. Prove that if $bc = |bc|$, then b and c are both positive, both negative or one is zero.
3. Suppose $|b| < |c|$. What conclusions can be drawn about inequalities between b and c?
4. If $|b| > |c|$, and $b < c$, what conclusions can be drawn about b and c?
5. Let the numbers b and c correspond to points B and C on the number line. Prove that

$$BC = |b - c| = |c - b|,$$

where BC denotes the distance between B and C.

6. Suppose

$$|b - c| = \frac{|c - d|}{3}.$$

If B and D are the points corresponding to the numbers b and d on the number line, what are the possibilities for the point C, corresponding to the number c?

Problem

1. Prove: $|a| \cdot |b| = |ab|$.

7.6 The structure of the set of integers

We now have the set of integers comprising the negative integers and the whole numbers. We have defined addition, subtraction and multiplication. But in this set we cannot always divide for there is still no solution in integers to the equation $3x = 5$, for example. If now we refer by comparison to the set of properties of the rational numbers listed in Section 5.8 we see that the set of integers has the following properties for addition and multiplication:

$$(1)\quad \left\{\begin{array}{l}\text{Closure (including that for } I^* \text{ under multiplication),}\\ \text{Commutativity, associativity, existence of identity elements,}\\ \text{Distributive property.}\end{array}\right.$$

It also is true that every element has an inverse element for addition but, in contrast to the set of non-negative rational numbers, it has in general no inverse element for multiplication. Thus for this set we have a group under addition but not under multiplication. The group is abelian.

For multiplication, in place of the existence of an inverse, the set of integers has a weaker but important property, namely the cancellation property. That is

$$ab = cb,\ b \neq 0 \text{ implies } a = c$$

for all integers a, b, c. To show this note that $ab = cb$ is equivalent to $ab - cb = 0$. By the distributive property we have

$$(a - c)b = 0.$$

But the set of non-zero integers is closed under multiplication. Hence the product can be zero only if one of its members is zero. But we have assumed that b is not zero. Hence $a - c = 0$, or $a = c$. The desired property is proved.

There is a name for a system which has the properties (1), the existence of an inverse for addition and the cancellation property for multiplication. It is called an integral domain.

More precisely, an integral domain is a set of numbers or elements with two operations, addition and multiplication, for which the following properties hold:

1. For addition: closure, commutativity, associativity, the existence of an identity element, 0, and an additive inverse for each element of the set, the "opposite number." (In other words, the set is a group under addition.)

2. For multiplication: closure, associativity, commutativity and the existence of an identity element 1.

3. If the product of two numbers is zero, at least one of them must be zero.
4. The distributive property.

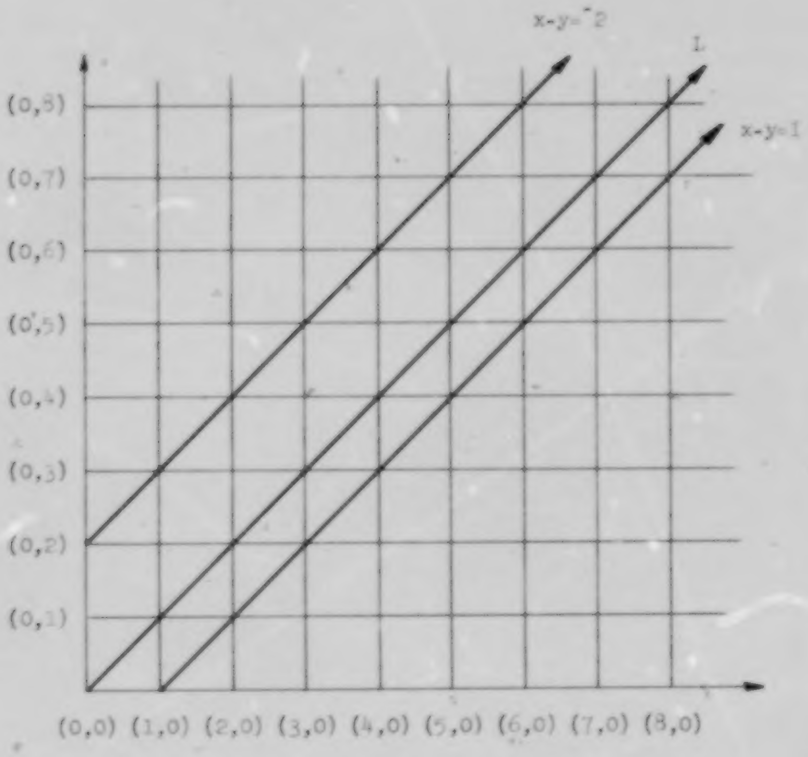
7.7 Negative integers as number pairs

Negative integers can be defined as number pairs of whole numbers as we did for rational numbers but with different definitions of equivalence, addition and multiplication. Mathematically this is the most satisfactory way to define negative integers but the motivation for the definitions associated with these number pairs lies in the properties which we developed above. In other words, this is another example of a mathematical structure built in accordance with our world as we see it.

Now then, we let the ordered pair of whole numbers (a,b) correspond to $a - b$ and have the following correspondences in the two notations for equality (or equivalence), addition and multiplication.

1.	$a - b = c - d$ if and only if $a + d = c + b$	$(a,b) = (c,d)$ if and only if $a + d = c + b$
2.	$(a - b) + (c - d) = (a + c) - (b + d)$	$(a,b) + (c,d) = (a + c,\ b + d)$
3.	$(a - b)(c - d) = (ac + bd) - (bc + ad)$	$(a,b)(c,d) = (ac + bd,\ bc + ad)$

To see the geometrical meeting of Property 1 in terms of the lattice points (x,y) where x and y are whole numbers, let L be the line through the points $(0,0)$ and $(1,1)$. All points on the line L will have the same difference, namely zero. Thus the lattice points on the line L correspond to the integer 0. The set of points (x,y) in which $x - y = 1$ is another line parallel to L; all the lattice points on this line correspond to the integer 1. Similarly the set of lattice points (x,y) in which $x - y = {}^{-}2$ correspond to the integer ${}^{-}2$ and lie on another line parallel to L. From this it is not hard to see in general that all the points corresponding to a fixed integer lie on a line parallel to L. (See figure).



Here the reader may want to refer back to our similar discussion for *non-negative* *rational* numbers in Section 5.11. There we showed that each such rational number corresponded to the set of lattice points on a ray through the origin. Here each integer corresponds to the set of lattice points on a ray which is parallel to L, the line through the origin and the point (1,1).

For Property 2, the student should try out the figures for a few values of a, b, c, d and try to form his own conclusions. In particular he should look at the figure formed by connecting by line segments the points (a,b) and (c,d) first with the origin and second with the sum $(a + c, b + d)$. This is proposed in a problem below and the answer given at the end of the book.

There does not seem to be a simple geometrical interpretation for multiplication of pairs according to the above definition.

Exercises

1. Let (a,b), (c,b), and (c,d) be the coordinates of points A, B and C. As in Exercise 5 for Section 7.5, or by other means, show the following:

$$AB = |a - c|, \ BC = |b - d|.$$

2. Use the results of the previous exercise to find the following formula for the distance between A and C.

$$AC = \sqrt{(a - c)^2 + (b - d)^2}$$

3. Use the formula developed in Exercise 2 to find the distance between the points (4,5) and (-2,6).

4. Suppose, in Exercise 1, (a,b) and (c,d) correspond to the same number, that is: $a + d = b + c$. Then find a relationship between the distances AB and BC.

5. In the correspondence between number pairs and integers given in this section, which number pairs correspond to the number 1, which to -1 and which to zero? Show that if (x,y) is a number pair corresponding to 1, then

$$(a,b)(x,y) = (a,b)$$

for every number pair (a,b). Show also that if (z,t) is a number pair corresponding to zero, then:

$$(a,b)(z,t) = (z,t).$$

Problems

1. Develop a method for finding the line which corresponds to a given integer, as described above.

2. Find a geometrical meaning for the sum of two number pairs as defined above.

3. Show that equlity of number pairs as defined above is an equivalence relationship in the sense of Section 5.4.

7.8 Negative rational numbers

Once the negative integers have been understood, it is not a very difficult step to pass on to the negative rational numbers. For instance, what should ${}^{-}\left(\frac{a}{b}\right)$ be? It should have the property that when we add it to $\frac{a}{b}$ we get zero. But

$$\frac{{}^{-}a}{b}+\frac{a}{b}=\frac{{}^{-}a+a}{b}=\frac{0}{b}=0.$$

Thus $\frac{{}^{-}a}{b}$ is the opposite of $\frac{a}{b}$ and we have

$${}^{-}\left(\frac{a}{b}\right)=\frac{{}^{-}a}{b}.$$

Also

$$\frac{{}^{-}a}{b}=\frac{{}^{-}1}{{}^{-}1}\cdot\frac{{}^{-}a}{b}=\frac{({}^{-}1)({}^{-}a)}{({}^{-}1)\ b}=\frac{a}{{}^{-}b}.$$

So we have

$${}^{-}\left(\frac{a}{b}\right)=\frac{{}^{-}a}{b}=\frac{a}{{}^{-}b}.$$

With the negative rational numbers we can complete the number line for all rational numbers.

7.9 Properties of the rational number system

Now we have the rational numbers. We defined addition and multiplication so that we would have certain selected properties of commutativity, associativity, distributivity and additive and (except for zero) multiplicative inverses. If we were to be systematic we would check to see that all these properties indeed hold. But in this book we prefer to take them for granted. We should, however list them. The letters below stand for rational numbers:

1. Closure: $a+b$ and ab are rational numbers.
2. Commutativity: $a+b=b+a$ and $ab=ba$.
3. Associativity: $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
4. Existence of Identity Element: For addition: $a+0=0+a=a$. For multiplication: $a\cdot 1=1\cdot a=a$.
5. Existence of an Inverse except for Division by Zero:
   For addition: for each a there is a ${}^{-}a$ such that
   $$a+{}^{-}a={}^{-}a+a=0.$$
   For multiplication: for each $a\neq 0$ there is a $\frac{1}{a}$ (sometimes written a^{-1} such that $\left(\frac{1}{a}\right)a=a\left(\frac{1}{a}\right)=1$.

6. The distributive property:

$$a(b + c) = ab + ac, \quad (b + c)a = ba + ca.$$

There are other systems which have all these properties, as we shall see in later chapters. Each such system is called a field. (In some books it is not assumed that for a field multiplication is commutative, but all the other properties are assumed.)

A short way of describing a field would be to call it a system which is closed under addition, subtraction, multiplication and division except by zero and for which the distributive property holds.

We have developed in turn four number systems: the whole numbers I', the nonnegative rational numbers R', the integers I and the rational numbers R. In all of these we had for multiplication and addition closure, commutativity, associativity, existence of an identity and the distributive property. The difference was in the existence of an inverse. We can exhibit this in the following table:

System	Existence of an inverse	
	for addition	for multiplication except for zero
I', the whole numbers	no	no
R', the nonnegative rationals	no	yes
I, the integers	yes	no
R, the rational numbers	yes	yes

However in all cases the cancellation and well-defined properties for addition and multiplication hold. Whenever we have an inverse the cancellation property must hold since we can add or multiply the inverse. But the cancellation property can hold when the inverse does not exist.

Problem

1. Prove that in a field the cancellation properties for addition and multiplication must hold, as a consequence of the well-defined property.

7.10 The rational numbers as number pairs of integers

Just as in Section 5.11 we considered expressing the positive rational numbers as ordered pairs of whole numbers, so we can consider the set of rational numbers as ordered pairs of integers. Here instead of having two perpendicular rays, we have two number lines perpendicular at the point (0,0). The lattice points will then be represented by the pairs (a,b) where a and b are integers. Each lattice point (a,b) will correspond to a rational number $\frac{a}{b}$, if $b \neq 0$. All the pairs corresponding to the same rational number will be on a line through the point (0,0) called the origin. It is an important fact that from any integral domain, D, a field may be constructed by means of ordered pairs of numbers of D just as we constructed the rational numbers from the integral domain of integers.

7.11 Inequalities

Whether we are considering the set I of integers or the set R of rational numbers, we can divide the set into three classes:

P: the set of positive numbers (integers or rational numbers)

O: the number zero

N: the set of negative numbers (negative integers or negative rational numbers)

These sets are categorical and disjoint. That is, every number in I or in R is in one of these sets and no number is in two of them. There are several important properties of this classification:

1. The set P is closed under multiplication and addition. That is, the sum of two positive numbers is positive and their product also.
2. If x is any number in I or R except zero, either x or ^-x is in P.
3. The product of two numbers in N is in P.
4. The product of a number in N and a number in P is in N.

Notice that the set N is closed under addition, since the sum of two negative numbers is negative. But it is not closed under multiplication since the product of two negative numbers is positive.

Recall also that in I and R, the equation $a + x = b$ is always solvable.

With this preparation we can give a definition of inequality.

Definition: Let a and b be two numbers in I or in R, and let x be the solution of $a + x = b$. Then

1. If x is in P, that is, if x is positive, we say that b is greater than a and write $b > a$.

2. If $x = 0$, then $a = b$.

3. If x is in N, that is, if x is negative, we say that b is less than a and write $b < a$.

Notice that the disjointness of the sets P, O and N implies that if b and a are any two numbers in I or in R exactly one of the following is true:

$$b > a,\ b = a,\ b < a.$$

Furthermore, if $a + x = b$ we can, by the well-defined property for addition add ^{-}x to both sides of the equation and have

$$b + {}^{-}x = (a + x) + {}^{-}x = a + (x + {}^{-}x) = a + 0 = a.$$

Thus $b + {}^{-}x = a$.

Now by the second property of P above, we have

If $b > a$, then x is positive, ^{-}x is negative and $a < b$.

If $b < a$, then x is negative, ^{-}x is positive and $a > b$.

So, as we know, $b > a$ and $a < b$ mean the same thing and $b < a$ and $a > b$ also mean the same thing.

For inequalities the well-defined and cancellation properties which we considered for equalities take on a new significance. First consider them for addition. They are, you will recall,

$$a = b \text{ implies } a + c = b + c$$
$$a + c = b + c \text{ implies } a = b.$$

For sets like I and R which are closed under subtraction, these two conditions are equivalent, for to get the second from the first we need only make the following replacements

$$a \longleftrightarrow a + c \qquad b \longleftrightarrow b + c \qquad c \longleftrightarrow {}^{-}c.$$

And a similar replacement will allow us to deduce the former from the latter. So we need consider only:

The well-defined property of addition for equality:

$$a = b \text{ implies } a + c = b + c.$$

The corresponding property for inequality is

$$a < b \text{ implies } a + c < b + c.$$

The property for inequality follows immediately from that for equality since $a + x = b$ implies $(a + c) + x = b + c$ and if x is positive in one equation it is positive in the other, if x is negative in one it is negative in the other. That is, if x is positive both $a < b$ and $a + c < b + c$. While if x is negative both $a > b$ and $a + c > b + c$. So the well-defined property for addition holds for inequalities just as for equalities.

For multiplication the situation is somewhat more complex. Recall that the two properties are:

Well-defined property for multiplication: $a = b$ implies $ac = bc$.

Cancellation property for multiplication: $ac = bc$, implies $a = b$ if $c \neq 0$.

Both of these properties hold in the set I of integers and in the set R of rational numbers but only in R are they equivalent since only in R is there a multiplicative inverse. So we consider the properties separately.

Thus we have the following question for a, b, c three numbers in I or R. Does $a < b$ imply $ac < bc$?

Now $a < b$ means $a + x = b$ where x is a positive number. So, by the well-defined property for multiplication and equality we have

$$ac + xc = bc.$$

The relationship between ac and bc all depends on whether xc is positive, zero or negative. There are, in fact three possibilities, from the properties of the sets P and N:

1. If c is positive, xc is positive and $ac < bc$.
2. If $c = 0$, $xc = 0$ and $ac = bc$.
3. If c is negative, xc is negative and $ac > bc$.

Thus we have: If $a < b$ then

1. if c is positive, $ac < bc$
2. if $c = 0$, then $ac = bc$
3. if c is negative, $ac > bc$.

It is easy to show by interchanging a and b, that the corresponding results hold when a is greater than b. We can phrase these results as follows:

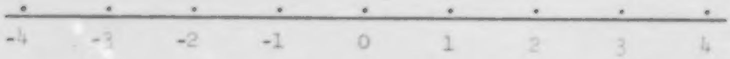
Multiplying the numbers on both sides of an inequality by a positive number preserves the direction of the inequality and multiplying by a negative number reverses the direction of the inequality. We call this the well-defined property of multiplication for inequality.

It can be shown similarly that the same situation exists with respect to the cancellation property. We state the result and leave the proof as an exercise:

if $ac < bc$ then $a < b$ if c is positive and $a > b$ if c is negative

if $ac > bc$ then $a > b$ if c is positive and $a < b$ if c is negative.

There is no problem in determining which of two integers is the greater if we merely consider the number line as follows:



Thus any number is less than another number if the point which corresponds to it is to the left of the point which corresponds to the other. Thus if two numbers are to the right of the point 0, the point closer to the point 0 represents the lesser number. If the two points are on the opposite side of the point 0, the number corresponding to the point on the left of 0 is the lesser. If both points are on the left, the one farther from 0 corresponds to the lesser number. Thus, in the last case, $^-5$ is less than $^-3$ because $^-5 + 2 = {}^-3$. Since the distance between 0 and a point is the absolute value of the number corresponding to the point, we can state these facts in terms of absolute value although it is not clear to the author how much this statement contributes to the understanding of the situation:

1. If a and b are in P, $a < b$ if and only if $|a| < |b|$.
2. If a is in N and b is in P, then $a < b$.
3. If a and b are in N, then $a < b$ if and only if $|a| > |b|$.

For instance, the numbers, 3 and 5 are in P and are the same as their absolute values. So of course $3 < 5$ if and only if $|3| < |5|$. Any negative number is less than any positive number. Finally, $^-5$ is less than $^-3$ since $|^-5| = 5$, $|^-3| = 3$ and $5 > 3$.

For the rational numbers we have to exercise a little care. Here the reader is referred to Section 5.9. We could use the same method here as there if we specified that the denominators of the fractions were to be positive. But a slightly more efficient and somewhat more informative way to deal with the matter at this stage is to build directly on the properties above. Suppose we have two rational numbers $\frac{r}{s}$ and $\frac{t}{u}$ and we wish to determine which is the greater. We learned above from the well-defined and cancellation properties of multiplication and inequalities,

$$a < b \text{ is equivalent to } ac < bc \text{ if } c \text{ is positive.}$$

So, taking $a = \frac{r}{s}$, $b = \frac{t}{u}$ and $c = su$ we have

$$\frac{r}{s} < \frac{t}{u} \text{ is equivalent to } (\frac{r}{s})su < (\frac{t}{u})su$$

provided that su is positive. But $(\frac{r}{s})su = ru$ and $(\frac{t}{u})su = ts$. So we have

$$\frac{r}{s} < \frac{t}{u} \text{ is equivalent to } ru < ts,$$

provided that su is positive. But since $\frac{r}{s} = \frac{^-r}{^-s}$, we can write any rational number as a fraction whose denominator is a positive integer. This means that if the two fractions are written so that the denominators are positive (or, for that matter, both negative),

$$\frac{r}{s} < \frac{t}{u} \text{ if } ru < ts \text{ and if } ru < ts, \text{ then } \frac{r}{s} < \frac{t}{u}.$$

For example, suppose we wish to compare the fractions

$$\frac{^-2}{3} \text{ and } \frac{5}{^-7}.$$

We write the second fraction in the form $\frac{^-5}{7}$ and see that

$$^-14 > {}^-15 \text{ implies } \frac{^-2}{3} > \frac{5}{^-7} .$$

Of course if one fraction represents a negative number and the other a positive number, we do not need any such complex means of comparison.

Problems

1. Why are the well-defined and cancellation properties of multiplication for equality equivalent in the set of rational numbers? That is, why does each imply the other?

2. Prove the cancellation property of multiplication for inequality in the set of rational numbers.

3. Refer to the definition of a group in Section 5.8 and state which of the following form a group under multiplication:
   a. The set, P, of positive integers.
   b. The set, P', of positive rational numbers.
   c. The set, N, of negative integers.
   d. The set, N', of negative rational numbers.
   e. The union of P and N, that is, all non-zero integers.
   f. The union of P' and N', that is, all non-zero rational numbers.
   g. The set of whole numbers.

7.12 Betweenness

In Section 3.2 we discussed betweenness for the set of whole numbers. Let us carry this further for the set of rational numbers. Referring to the number line we see that a point is between two others if the number which corresponds to it is between the numbers corresponding to the other two points. In other words, let A, B, C be three points on the number line and a, b, c the numbers corresponding to them, then B is between A and C if one of the following sets of inequalities holds:

$$a < b < c \text{ or } a > b > c.$$

We say that the number b is between the numbers a and c. It is not hard to prove the following for rational numbers a, b, c, d:

1. If b is between a and c, then $b + d$ is between $a + d$ and $c + d$.
2. If b is between a and c, then bd is between ad and cd if $d \neq 0$.

The first is left as a problem. To prove the second, suppose

$$a < b < c.$$

Then if d is positive, we have from properties described above:

$$ad < bd < cd.$$

On the other hand, if d is negative,

$$ad > bd > cd.$$

In both cases bd is between ad and cd and our result is proved.

Exercises

1. Let b, c and d be numbers corresponding to three points on the number line. Prove that if the point corresponding to c is between the points corresponding to the other two numbers then
$$|b - c| + |c - d| = |b - d| .$$
2. Show that if b, c and d are three rational numbers such that
$$|b - c| + |c - d| = |b - d|$$
then the point corresponding to c on the number line is between the points corresponding to the other two numbers. (See Section 3.2).
3. Let b, c and d be three rational numbers. Show that one of the following is the sum of the other two:
$$|b - c|, |b - d|, |d - c|.$$
(See Section 3.2).

Problems

1. Prove Property 1 of betweenness above.
2. What is needed to complete the proof for Property 2?
3. If a is between b and c and c is between a and d, prove that a and c are between b and d.

7.13 The triangle inequality

Now we discuss two important applications of the concept of absolute value and the properties of inequalities. The first theorem is quite easy to prove:

Theorem 1. If $a > b > 0$, then $a^2 > b^2$.

Proof: Since $a > b$ and $a > 0$, then $a^2 > ba$ from the well-defined property of multiplication for inequality. Similarly, $a > b$ and $b > 0$ implies $ba > b^2$. The transitive property of inequality then completes the proof.

Theorem 2. (The triangle inequality). If a and b are rational numbers, then

$$|a + b| \leq |a| + |b|.$$

Proof: We know that $|a|^2 = a^2$ since either $|a| = a$ in which case $|a|^2 = a^2$ or $|a| = -a$ in which case $|a|^2 = (-a)^2 = a^2$. Similarly $|b|^2 = b^2$ and $|a + b|^2 = (a + b)^2$. Now suppose that the conclusion of the theorem were false. Then we would have

$$|a + b| > |a| + |b|,$$

and it would follow, by Theorem 1, that

$$(1) \qquad (a + b)^2 > (|a| + |b|)^2 = a^2 + b^2 + 2|ab|.$$

since (see Section 7.5) $|a| \cdot |b| = |ab|$. On the other hand, we know that

$$(a + b)^2 = a^2 + b^2 + 2ab.$$

There are then two possibilities which we consider in turn:

i) $ab \geq 0$. In that case, the inequality, (1), becomes

$$a^2 + b^2 + 2ab > a^2 + b^2 + 2ab$$

which is false.

ii) If $ab < 0$, we would have

$$a^2 + b^2 > a^2 + b^2 + 2ab > a^2 + b^2 + 2|ab| > a^2 + b^2$$

which is also impossible. Hence our assumption that the inequality of the theorem is false leads to a contradiction. This completes the proof.

Though this theorem was proved for rational numbers we used only the properties of Section 7.11 together with the properties of a field and hence any system which has these properties will also be a system in which the triangle inequality holds. In particular, the real numbers which are dealt with in the next chapter have these properties and the triangle inequality holds for them also. In fact, with a definition of absolute value for complex numbers, it holds for these as well.

The reason for the name "triangle inequality" lies in its connection with vectors and the discussion of complex numbers in Appendix III. At this point suffice it to say that geometrically, the triangle inequality is this:

> If A, B and C are three points, then the sum of any two of the distances AB, AC, BC is not less than the third. If the sum of two is a third, the points are collinear--otherwise the three points are the vertices of a triangle.

Exercises

1. (See Problem 4). Under what conditions is the following true:

$$|a - b| = |a| + |b| ?$$

2. If a and b are positive rational numbers with $a > b$, does it always follow that:

$$-\frac{1}{a} > -\frac{1}{b} ?$$

3. Let a be a positive rational number and c a rational number such that $a + c > 0$. Prove the following:

$$(a + c) + \frac{a^2}{a + c} \geq 2a.$$

and the equality holds only if $c = 0$.

4. Using Exercise 3 or by other means prove that if b is a rational number different from zero and a a positive rational number, then

$$\left|b + \frac{a^2}{b}\right| \geq 2a.$$

Apply this to an improvement of the result of Problem 5.

Problems

1. If in Theorem 1 we require only that $a > b$ and a and b be different from zero, does the conclusion necessarily hold?

2. If a and b are rational numbers and $a > b$, does it always follow that $a^3 > b^3$?

3. Prove for rational numbers a and b: $|a - b| \geq \big||a| - |b|\big|$.

4. Prove that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

5. Prove that if $b \neq 0$, then

$$\left|b + \frac{1}{b}\right| > 1.$$

References

6 (Chapter 3), 8 (Chapter 5), 11, 12 (Chapters 3, 5), 21 (Chapters 2, 3, 4), 22 (Chapter 12).

Chapter 8

THE REAL NUMBERS

8.1 Introduction

Though there is probably not much to be said about the real numbers to a student in junior high school, he can scarcely fail to meet some numbers which are not rational ones. The number π is not rational, though this is not easy to prove. The number $\sqrt{2}$ is not rational and the proof can be found almost anywhere. Just for variety we give one which is a little different from the usual one. Suppose 2 were the square of a rational number. Then we would have

$$2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$

that is

$$(1) \qquad 2b^2 = a^2.$$

Consider the factorization of b into prime factors: $b = pqr \ldots t$. Then $b^2 = p^2q^2 \ldots t^2$. So if 2 is a factor of b it appears an even number of times as a factor of b^2 (note that zero is an even integer). Thus in $2b^2$ it appears an odd number of times as a factor. But in a^2, 2 appears an even number of times as a factor. It is impossible for 2 to appear an even number of times on one side of (1) and an odd number of times on the other since both sides represent the same number. Thus our supposition that (1) is possible is false and there is no rational number whose square is 2. We also noted that the decimal

$$.1010010001\ldots$$

cannot represent a rational number. All these considerations point to the need, though a somewhat more sophisticated one, for numbers beyond the rational numbers.

8.2 Definition of a real number

Up to this point we could always have extended our number system by considering pairs of numbers of the previous system. But an argument in Appendix II of this book shows that this will not suffice for passing from the rational numbers to the real numbers. In fact, triples, quadruples, etc., of numbers would not do either. One has to use an entirely new approach.

Probably the simplest way to think of a real number is to consider it to be any number which can be "represented" by a decimal. But though this may do to give one a vague idea, yet many difficulties are bound up in the word "represented." For one thing, a number has to exist before it can be represented. One may also think of a real number as one which can be used to represent the distance between any two points, but it is easier to define distance in terms of real numbers than the other way around. A rigorous definition of a real number is much too difficult for us here but we can perhaps give some idea of how a rigorous definition could be made. Let us approach our definition by the consideration of two examples:

First of all, consider the unending decimal:

$$.33333\ldots \quad .$$

We know that this is the decimal expansion for one-third, but let us look at some of its characteristics. It can be considered as a sequence of numbers as follows:

$$L_1 = .3,\ L_2 = .33,\ L_3 = .333,\ \ldots,\ L_n = .333\ldots3,\ \ldots$$

where L_n contains n 3's. We can associate this sequence with another:

$$U_1 = .4,\ U_2 = .34,\ U_3 = .334,\ \ldots,\ U_n = .333\ldots34,\ \ldots$$

where U_n has $(n - 1)$ 3's and its last digit is a 4. We got each U from its corresponding L by adding an appropriate power of 10 as follows:

$$U_1 = L_1 + .1 = L_1 + 10^{-1},\quad U_2 = L_2 + .01 = L_2 + 10^{-2},$$

$$U_3 = L_3 + .001 = L_3 + 10^{-3},\quad U_4 = L_4 + .0001 = L_4 + 10^{-4}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$U_n = L_n + .000\ldots01 = L_n + 10^{-n}.$$

In other words, for each natural number n, we have

$$U_n - L_n = 10^{-n} = \frac{1}{10^n}\ . \tag{1}$$

These two sequences of numbers have the following four properties:

1. The L's form a nondecreasing sequence, that is, each is greater than or equal to its predecessor. In notation:

$$L_1 \leq L_2 \leq L_3 \leq \dots \leq L_n \leq \dots .$$

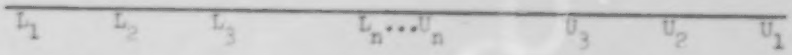
2. The U's form a nonincreasing sequence, that is, each is less than or equal to its predecessor. In notation:

$$U_1 \geq U_2 \geq U_3 \geq \dots \geq U_n \geq \dots .$$

3. Every U is greater than every L.

4. As n becomes larger and larger, the difference $U_n - L_n$ approaches zero.

These properties can be represented graphically as follows:



(We have not tried to represent this to scale.) The first two properties are apparent. Consider Property 3. Equation (1) shows us that L_n is less than U_n. But all the preceding L's are less than L_n and hence less than U_n which in turn is less than all the following U's. In notation, our argument is:

$$L_i \leq L_n < U_n \leq U_j \text{ , if } i \text{ and } j \text{ are less than } n.$$

Finally, Property 4 follows from the Equation (1) and the fact that $\frac{1}{10^n}$ approaches zero as n becomes larger and larger. More precisely, by choosing n large enough we can make the difference $U_n - L_n$ as small as we please.

So associated with the decimal .333 ... we have two sequences with the properties listed above. There is just one number, one-third, which is greater than or equal to all the L's and less than or equal to all the U's. In other words if a number k has the property that

$$L_n \leq k \leq U_n$$

for all values of n, then k must be one-third. This is the number which is defined by the decimal. (In this case we did not need the qualification "or equal to" but there are cases in which we will need it.)

Now consider the decimal associated with $\sqrt{2}$. We proceed to construct two sequences having the four properties above

Step 1. We know 1^2 is less than 2 and $2^2 = 4$ is greater than 2, so we write

$$L_0 = 1 \text{ and } U_0 = 2.$$

Step 2. We know $1.4^2 = 1.96$ is less than 2 and $1.5^2 = 2.25$ is greater than 2. So we choose

$$L_1 = 1.4 \text{ and } U_1 = 1.5 .$$

Step 3. Similarly $1.41^2 = 1.9881$ which is less than 2 and $1.42^2 = 2.0164$ which is greater and so we write:

$$L_2 = 1.41 \text{ and } U_2 = 1.42 .$$

Step 4. To find L_3 we could try in succession the squares of 1.410, 1.411, 1.412, 1.413, 1.414... until we find the greatest one which is less than 2. This process would have to stop because we know that the square of 1.42 is greater than 2. It turns out that

$$L_3 = 1.414 \text{ and } U_3 = 1.415 .$$

We can check our calculations from a table of square roots where the square root of 2 is listed as 1.4142 to four decimal places, which implies that 1.4141^2 is less than 2 and 1.4143^2 is greater than 2. At any rate we have found the first four terms of the two sequences:

$$L_0 = 1,\ L_1 = 1.4,\ L_2 = 1.41,\ L_3 = 1.414$$

$$U_0 = 2,\ U_1 = 1.5,\ U_2 = 1.42,\ U_3 = 1.415$$

We could carry this computation further along the lines indicated above. But the important thing to see is that we have so constructed the two sequences that the four properties hold as well as Property (i). In addition, we have so constructed our sequences that the square of every L is less than 2 and the square of every U is greater than 2. (This could be used to establish Property 3.)

In the case of the decimal .333... there was a rational number which was greater than or equal to all the L's and less than or equal to all the U's. We now show

Theorem 1. If there were a number k with the property that it is greater than or equal to all the L's in the sequence above associated with the square root of 2 and less than or equal to all the U's, then its square would have to be 2.

Notice that when we have established this, we will have shown that k could not be a rational number since we have shown that there is no rational number whose square is 2. Now, to begin the proof, we have

$$L_n \leq k \leq U_n$$

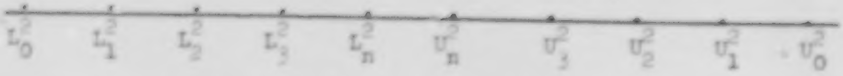
for all integers n. Then (see Section 7.13) we have

$$L_n^2 \leq k^2 \leq U_n^2$$

for all integers n. But, from the construction of the sequences we have also

$$L_n^2 \leq 2 \leq U_n^2 .$$

Our results up to this point are shown in the following figure:



The both numbers 2 and k^2 are between L_n^2 and U_n^2 and hence the distance between 2 and k^2 must be less than or equal to the distance between L_n^2 and U_n^2. In notation this is

$$U_n^2 - L_n^2 \geq |2 - k^2|.$$

But

$$U_n^2 - L_n^2 = (U_n - L_n)(U_n + L_n) \leq \frac{4}{10^n}$$

since $U_n - L_n = \frac{1}{10^n}$ and

$$U_n + L_n < U_n + U_n = 2U_n \leq 2U_0 = 4.$$

Thus, by choosing n large enough we can make the difference between 2 and k^2 as small as we please. This is impossible unless k^2 is equal to 2. This completes the proof of Theorem 1.

Thus we have shown that there is no rational number which is greater than or equal to all the L's and less than or equal to all the U's. This is an unsatisfactory state of affairs and so we create such a number just as we have created numbers before. Or, alternately, we might postulate the existence of such a number. We state this formally as a postulate:

The Completeness Postulate: Suppose we have two sequences of decimals L_n and U_n with the four properties listed above. Then there is a unique number which is greater than or equal to all the L's and less than or equal to all the U's.

Definition: The set of real numbers consists of all numbers postulated above.

Note: Actually the sequences L and U need not be decimals and it is true that property 4 implies that the word "unique" is superfluous by an argument like that which we used in the proof of Theorem 1. But the postulate and definition is sufficient for our purposes here.

Now we are in a position to give a more definite meaning to the phrase "represented by a decimal." Suppose we have a decimal which we write as follows:

$$.a_1a_2a_3 \ \ldots \ a_n \ \ldots$$

where the a's are the digits. If the decimal terminates, this just means that from a certain point on all the a's are zero. Then we write the series of L's:

$$L_1 = .a_1 \, , \; L_2 = .a_1a_2 \, , \; L_3 = .a_1a_2a_3 \, , \; \ldots \, , \; L_n = .a_1a_2a_3 \ldots a_n .$$

For all values of n we define

$$U_n = L_n + 10^{-n}$$

as before.

Now we want to show that these two sequences satisfy the four properties above. Property 1 is evident. (Notice that if a digit is zero, two successive L's may be equal.) Property 4 follows from the definition of U_n.

Property 2 may be intuitively evident, but it is better to give a formal proof. First we need some information about the sequence of L's. Since no digit a is greater than 9, we have the following sequence of inequalities:

$$\begin{aligned}
L_2 &\leq L_1 + .09 = L_1 + 9 \cdot 10^{-2} \\
L_3 &\leq L_2 + .009 = L_2 + 9 \cdot 10^{-3} \\
L_4 &\leq L_3 + .0009 = L_3 + 9 \cdot 10^{-4} \\
&\cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\
L_n &\leq L_{n-1} + 9 \cdot 10^{-n} = L_{n-1} + \frac{9}{10^n} \\
L_{n+1} &\leq L_n + \frac{9}{10^{n+1}} \, .
\end{aligned}$$

The last inequality then gives us:

$$L_{n+1} - L_n \le \frac{9}{10^{n+1}} .$$

Since $U_n = L_n + 10^{-n}$, this leads immediately to

$$U_{n+1} - U_n = L_{n+1} + \frac{1}{10^{n+1}} - L_n - \frac{1}{10^n}$$

$$= (L_{n+1} - L_n) + \frac{1}{10^{n+1}} - \frac{10}{10^{n+1}}$$

$$\le \frac{9}{10^{n+1}} - \frac{9}{10^{n+1}} = 0 .$$

Thus we have shown that $U_{n+1} - U_n \le 0$, that is $U_{n+1} \le U_n$, which establishes Property 2. Then, just as previously, Property 3 follows since

$$L_i \le L_n < U_n \le U_j$$

for all i and j less than n. Looking back over the argument we see that we began with a decimal between 0 and 1, but this was only a matter of convenience in notation. We could have utilized the same development with any decimal. Thus we have shown that from any decimal there may be constructed two sequences of U's and L's with the four properties. Then the number postulated above is the number which we say is associated with the decimal or represented by it. In turn, we can speak of the *decimal expansion* of a real number.

The reader may think that this is a long way around to come to something so obvious as the concept of a number represented by a decimal. The point is that along these lines one may define a real number and that this method places the real number in relation to the two sequences. Furthermore, the process also places the real numbers on the number line. Starting with this, one could prove that the *real numbers form a field which is ordered*. This, however, is too difficult to attempt here.

There are two kinds of real numbers: rational numbers and irrational numbers (those which are not rational). There would also be negative and positive real numbers. It is a little hard to show that between any two rational numbers there is an irrational number and between every two irrational numbers there is a rational number. (See Section 9.3). We can think of the real numbers as "filling up the number line," whatever that means.

The real numbers form an ordered set which has the same properties as were described for rational numbers in Section 7.11. One of the consequences of this is that the square of every non-zero real number is a positive number. An important consequence of this is that the equation $x^2 = -1$ has no solution in the set of real numbers. This lack can be taken care of in the creation of the complex numbers as discussed in Appendix 3, by considering ordered pairs of real numbers with certain prescribed properties. The complex numbers, however, do not have the properties of an ordered set.

Problems

1. Prove that there is no rational number whose square is 3 .

2. Prove that there is no rational number whose cube is 2 .

3. Prove that if r is a rational number different from zero and s is an irrational number, then rs is irrational.

4. If r and s are two rational numbers and $r < s$, show that

$$r + (s - r)(\frac{\sqrt{2}}{2})$$

is an irrational number between r and s.

5. Find a rational number between $\sqrt{2}$ and $\sqrt{3}$. Describe a method of finding a rational number between any two given irrational numbers.

6. Find the first five members of the sequence of U's and L's for each of the following: i) the decimal for $\frac{1}{11}$; ii) the decimal for $\frac{2}{5}$: iii) the decimal for π , which to five places is 3.14159.

7. There are two decimals which represent $\frac{2}{5}$: .40000... and .39999... . Find the sequences of L's and U's for each.

8. Consider the following two sequences, prove that they have the four properties described above, and find what number they define:

$$L_0 = \frac{1}{2}, \quad L_1 = \frac{2}{3}, \quad L_2 = \frac{3}{4}, \ . \ . \ . \ . \ , L_n = \frac{1 + n}{2 + n} .$$

$$U_0 = \frac{3}{2}, \quad U_1 = \frac{4}{3}, \quad U_2 = \frac{5}{4}, \ . \ . \ . \ . \ , U_n = \frac{3 + n}{2 + n} .$$

9. In Problem 8 above replace the two sequences by the following:

$$L_0 = \frac{1}{4}, \quad L_1 = \frac{2}{6}, \quad L_2 = \frac{3}{8}, \ . \ . \ . \ . \ , L_n = \frac{1 + n}{4 + 2n}.$$

$$U_0 = \frac{3}{4}, \quad U_1 = \frac{4}{6}, \quad U_2 = \frac{5}{8}, \ . \ . \ . \ . \ , U_n = \frac{3 + n}{4 + 2n} .$$

Exercises

1. Consider the exercises of Section 7.13. There it was assumed that the letters stood for rational numbers. Would the same results hold for any real numbers?

2. Suppose we consider the numeral system to the base seven. Recall that then .1 means one-seventh, .01 means one forty-ninth, etc. Could real numbers be defined in terms of these "decimals"? If so, how could this be done?

3. Give some examples of decimals which represent irrational numbers.

4. Let $t_0 = 1$, $t_1 = 1 - \frac{1}{2}$, $t_2 = 1 - \frac{1}{2} + \frac{1}{2^2}$, $t_3 = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3}$, ... ,

$$t_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \dots + \frac{(-1)^{n-1}}{2^{n-1}} + \frac{(-1)^n}{2^n} .$$

Then define two sequences of L's and U's as follows:

$$L_0 = 0 , \quad L_1 = t_1 , \quad L_2 = t_3 , \quad L_3 = t_5 , \ \dots , L_r = t_{2r-1}$$

$$U_0 = t_0 , \quad U_1 = t_2 , \quad U_2 = t_4 , \quad U_3 = t_6 , \ \dots , U_r = t_{2r} .$$

In words, the L's are the t's with the odd subscripts, and the U's are the t's with even subscripts. Show that these two sequences satisfy the four properties of this section. What number is represented?

References

6 (Chapters 3, 5), 12 (Chapter 6), 13 (Chapters 3 and 4).

Chapter 9

EQUATIONS AND GRAPHS

9.1 Introduction

We have been using glibly through this book equations like

$$a + b = b + a , \quad a + x = b .$$

The first was used to express a property of numbers; that is, if a and b are any two numbers (perhaps rational numbers), then $a + b = b + a$. For the second equation the point of view was quite different. We thought of the numbers a and b as given fixed numbers and wanted to find for what numbers x the equation would be true.

Traditionally an equation was "an indicated equality between two expressions." In most of the newer texts now it is thought of as a sentence composed of two "number phrases" with a verb phrase "is equal to." Mechanically there are two "sides" of an equation: that which is to the left of the equality sign and that which is to the right. These are the two "number phrases." Thus an equation is a statement that two numbers are equal. Sometimes the statement is true, and sometimes it is not.

We may have equations which involve numbers alone like:

$$3 + 5 = 6 , \quad 4 + 7 + 2 = 13 , \quad \frac{0}{0} = 1 .$$

The first of these is an equation which is not true. (Some people would prefer to say that it is not an equation, but in that case they would have a different definition of an equation.) The second is an equation which is true. The third is an equation which is not true because it does not have any meaning, in contrast to the first one which has a meaning.

We also have equations involving letters and numbers like:

$$a + 5 = 5 + a , \quad x = 7 ,$$

$$a + 1 = \frac{(a - 1)(a + 1)}{a - 1} , \quad 3x + 5 = 2 , \quad x + 1 = x .$$

The first of these is true for all numbers a (or if you like, all values of a). This equation and any other which is true, no matter what numbers are substituted for the letters, is called an identity. The second equation is true if the letter x is replaced by 7 but not otherwise. The third equation is true for all values of a except a = 1. The reason for the exception is that if a is 1, the right side becomes

$$\frac{0 \cdot 2}{0}, \text{ that is, } \frac{0}{0},$$

which, as we have noted before, has no meaning. It is general practice, though not very consistent, to call this equation an identity too. This requires a slight modification of the definition of identity as follows:

> An identity is an equation which is true for all substitutions of numbers for the letters for which both sides of the equation have meaning.

The equation $3x + 5 = 2$ is not true for any counting number nor for any non-negative rational number. It is true if $x = -1$. We call this value of x a solution of the equation. That is, any number for which the equation is true is said to "satisfy the equation" or is "a solution of the equation." From this point of view, if every number is a solution of the equation, it is an identity.

Finally the equation $x + 1 = x$ is not true for any number x. It has no solutions.

Equations which are not identities are often called conditional equations. The set of solutions of an equation is often called the solution set of the equation.

9.2 Solving equations

The process of finding all the numbers which satisfy an equation is called "solving the equation." Sometimes this can be done by inspection as for $x + 5 = 7$. We can tell by looking at it that $x = 2$ is a solution and only this number satisfies the equation. For more complicated equations the process consists in finding a succession of "equivalent" equations on the way to an equation whose solution is obvious. Two equations are said to be equivalent if they have the same solution set. For instance, the pair $x + 1 = 5$ and $x - 1 = 3$ are equivalent, since $x = 4$ is a common solution and they have no other. In fact, $x + 1 = x$ and $x + 2 = x$ are equivalent equations because neither has a solution; that is, their solution sets are the null set.

There are certain fundamental processes which we can be sure yield equivalent equations. If we confine ourselves to these we know that from step to step we do not change the solution sets, so that at the end the final equation will give us all and only the solutions of that with which we began. These stem from the well-defined and cancellation properties for addition and multiplication. Recall that for numbers they are the following:

1. For addition: if $a = b$, then $a + c = b + c$
   if $a + c = b + c$, then $a = b$
2. For multiplication: if $a = b$, then $ac = bc$
   if $ac = bc$ $\underline{\text{and } c \neq 0,}$ then $a = b$.

Then if the sides of an equation represent numbers and if we get another equation from the first by adding or subtracting any number to both sides or by multiplying by or dividing by any number different from zero, then the solution set of the first equation will be the same as that for the second.

Let us see how this works for a numerical equation. Consider

$$2x + 3 = 20.$$

Here we have in mind that we want to end with an equation like $x = b$ for some number b. So first we add $^-3$ to both sides and get

$$2x + 3 + {}^-3 = 20 + {}^-3$$

or

$$2x = 17 .$$

We know from Property 1 above that the solution set of this equation is the same as that of the given one. We would have had the same result had we subtracted 3 from both sides.

Next we divide both sides by 2 or multiply both sides by one-half to get

$$x = \frac{17}{2} .$$

This last equation has just one solution, $\frac{17}{2}$, and hence the first equation has the same solution. The purpose of checking this by substituting $\frac{17}{2}$ in the given equation is to find whether we made a mistake in the process. If we were sure that there were no mistake, we could be sure that what we finished with provided all the solutions.

As long as one is solving linear equalities only (like $ax + b = c$), one may confine himself to the processes of adding, subtracting, multiplying, and dividing by numbers (numbers different from zero in the last two processes) without danger of trouble.

In fact, whenever we keep to these processes we are on safe ground. When we use others, care is necessary. For instance, consider the equation:

(1) $$\frac{1}{x - 1} + \frac{1}{x - 2} = \frac{-1}{(x - 2)(x - 1)}$$

To eliminate the fractions, we should multiply by $(x - 2)(x - 1)$. Let us do this and explore the consequences. This yields:

$$\frac{(x - 2)(x - 1)}{x - 1} + \frac{(x - 2)(x - 1)}{x - 2} = \frac{-(x - 2)(x - 1)}{(x - 2)(x - 1)}$$

(2) $$x - 2 + x - 1 = -1 .$$

Now let us analyze what we have done. By the first part of Property 2 above, if (1) is true for some number x, then (2) is also true; in other words, every solution of equation (1) is a solution of equation (2). Another way to say this is that the solution set of (1) is contained in the solution set of (2).

Furthermore, we can obtain equation (1) from (2) by dividing by $(x - 2)(x - 1)$, and the second part of Property 2 above shows that, unless this divisor is zero, every solution of (2) is a solution of (1). Now

$$(x - 2)(x - 1) = 0$$

if and only if $x = 2$ or $x = 1$, since the product of two numbers is zero if and only if one of them is. Thus we have shown that, aside from $x = 2$ and $x = 1$ (which may be solutions of (2) but not (1)), the solutions of (1) and (2) are the same.

It remains to solve equation (2) and apply what we have noted. Equation (2) can be written

$$2x - 3 = -1 .$$

By Property 1 above it is equivalent to

$$2x = 2 ,$$

which from Property 2 is equivalent to $x = 1$. Hence the only solution of equation (2) is $x = 1$. This means that except for the possibility of $x = 1$, there can be no solutions of equation (1). But $x = 1$ is not a solution of (1) since for $x = 1$ the first denominator is zero. Hence (1) has no solutions; that is, the solution set of (1) is the null set.

For solving quadratic equations, the technique is quite different. Here one's object is to arrive at an equation like

$$(x - a)(x - b) = 0.$$

Then we would know that either $x - a = 0$ or $x - b = 0$, since the product of two numbers can be zero only when one of them is. Thus to solve

$$x^2 - 3x = 3x - 5$$

one finds an equivalent equation in which the number 0 alone appears on the right side of the equality sign. To this end we subtract $3x - 5$ from both sides to get

$$x^2 - 6x + 5 = 0.$$

By multiplication we can verify that $x^2 - 6x + 5 = (x - 5)(x - 1)$, and thus the solutions of our given equation are the solutions of $x - 5 = 0$ together with those of $x - 1 = 0$. That is, $x = 5$ and $x = 1$. These are the only solutions.

Exercises

1. Solve the following equations for x:

   a) $3x + 5 = 2$ b) $7 - 2x = 5 + 5x$

   c) $\frac{2}{x} = 3$ d) $x^2 = 4$.

2. Consider the following "method of solution" of the equation

   $$(x - 10)(x - 8) = 3 .$$

   Since $3 = 1 \cdot 3$, we have two possibilities:

   $$x - 10 = 1 \quad \text{and} \quad x - 8 = 3$$

   or

   $$x - 10 = 3 \quad \text{and} \quad x - 8 = 1 .$$

   The first pair gives $x = 11$ and $x = 11$, which is a solution since

   $$(11 - 10)(11 - 8) = 3 .$$

   But the second pair gives $x = 13$ and $x = 9$, both of which cannot be true. Yet $x = 7$ is a solution of the given equation--this did not appear by the method given. What is wrong, or what alteration should be made?

3. Can the methods used in the previous exercise be used to solve:

   $$(x - 2)(x - 4) = 5 ?$$

   Why or why not?

4. Show that the equation of Exercise 3 is equivalent to each of the following:

   $$x^2 - 6x + 3 = 0 ,$$

   $$x^2 - 6x + 9 = 6 ,$$

   $$(x - 3)^2 = 6 ,$$

   $$x - 3 = \sqrt{6} \quad \text{or} \quad x - 3 = -\sqrt{6}$$

   $$x = 3 + \sqrt{6} \quad \text{or} \quad x = 3 - \sqrt{6} .$$

Problems

1. Show that equivalence of equations is an equivalence relationship in the sense of Section 5.4.

2. Consider the equation

$$(x - 2)(x - 1) = (x - 1) .$$

To solve this, one may divide both sides by $(x - 1)$ and get

$$x - 2 = 1 ,$$

or $x = 3$. Is $x = 3$ a solution of the given equation? Are there others? What pairs of equations are equivalent? Give reasons for your answers.

3. Find all the solutions of the equation

$$\frac{1}{x - 1} + \frac{1}{(x - 1)(x - 2)} = 1 .$$

Explain which pairs of equations in the process of your solution are equivalent.

9.3 Solving inequalities

For inequalities we could, with only slight alterations, use the same discussion as appeared for equations in the first section of this chapter. We have solution sets of inequalities and equivalent inequalities just as for equations. The basic techniques are the same except that at one point there is a difference. Again it is based on the well-defined and cancellation properties. We state them again here for reference and comparison.

1. For addition: If $a < b$, then $a + c < b + c$.
   If $a + c < b + c$, then $a < b$.
2. For multiplication: If $a < b$, then $ac < bc$, provided that $c > 0$.
   If $ac < bc$, then $a < b$, provided that $c > 0$.
   If in either of the cases for multiplication $c < 0$, then the second inequality is reversed.

We thus must have regard for the sign of any multiplier. (Not only are inequalities important in themselves, but they are also important as a means of emphasizing the properties of manipulation of equations.) Let us illustrate the process by solving an inequality in two ways:

$$^{-}2x + 3 < 7.$$

Method I. Subtract 3 from both sides to get the equivalent inequality $^{-}2x < 4$. To find the solution we would want to divide both sides by $^{-}2$. This is negative. Hence we reverse the direction of the inequality and get

$$x > {}^{-}2 .$$

This is equivalent to the given inequality which, therefore, has the solution $x > {}^{-}2$.

Method II. Add 2x to both sides to get the inequality: $3 < 7 + 2x$. Subtract 7 from both sides to get $^{-}4 < 2x$. Here we want to divide by 2, which is positive. This leaves unaltered the direction of the inequality. and we have as our final equivalent inequality

$$^{-}2 < x .$$

Though in this book we shall be concerned for the most part only with linear inequalities, we might look at the inequality corresponding to one of the problems in the previous section:

$$(x - 2)(x - 1) > (x - 1).$$

Here there are really two cases to consider.

Case 1. If $x - 1 > 0$, that is $x > 1$. In this case the given inequality is equivalent to $x - 2 > 1$; that is, $x > 3$. But here if $x > 3$, certainly $x > 1$. Hence our only condition is $x > 3$.

Case 2. If $x - 1 < 0$, that is $x < 1$. In this case the given inequality is equivalent to $x - 2 < 1$, that is $x < 3$. Since $x < 1$ implies $x < 3$, we see that the given inequality holds if $x < 1$.

The conclusion combining the two cases is

$$x < 1 \ \underline{\text{or}} \ x > 3.$$

As an application of these techniques we recall and solve a postponed problem from Section 5.10. There, using the second definition of density, we stated the following result which we now give as a theorem.

Theorem: Between any two real numbers there is a rational number.
Proof: This could be shown using decimal expansions. Though notationally this is rather difficult to do, in any numerical case the technique is clear. For instance, if two real numbers were

$$1.41459\ldots \quad \text{and} \quad 1.4151\ldots ,$$

a rational number between them is 1.4146.

We give here another proof because though it is somewhat more sophisticated, it involves some important ideas and can be made much more definite. First we prove a lemma which is an extension of a result in Section 4.2. (An auxiliary result is often called a "lemma.")

Lemma. Let s and t be two positive real numbers; then there is an integer q and a real number r such that

$$s = qt + r, \ 0 \leq r < t .$$

Proof: Notice first what this affirms geometrically. We have two points s and t on the number line and to the right of the point 0. If t is to the right of s, that is, $t > s$, we take $q = 0$ and $r = s$. If $t \leq s$, we lay off distances t on the line repeatedly until we have the greatest multiple of t less than s. This multiple of t is q and $s - qt = r$. In a way we divide s by t and write the remainder just as if t were a whole number.

To make this more precise, we consider the integral multiples of t:

$$t \ , \ 2t \ , \ 3t \ , \ 4t \ , \ \ldots \ .$$

Since by choosing the integer n large enough, we can make nt as large as we please, there will be some integer n such that* $nt > s$. Let m be the least such value of n. Then choose $q = m - 1$ and $r = s - (m - 1)t$. Then

$$s = (m - 1)t + r .$$

It remains to verify the conditions on r. In the first place, $s < mt$ implies

$$\begin{aligned} s = (m - 1)t + r &< mt \\ mt - t + r &< mt \\ -t + r &< 0 \\ r &< t , \end{aligned}$$

which is part of the condition. It remains to show that r is non-negative. Suppose $r < 0$. Then

$$\begin{aligned} r = s - (m - 1)t &< 0 \\ s &< (m - 1)t, \end{aligned}$$

which contradicts our choice of m to be the least multiple of t greater than s. This completes the proof of the lemma.

---

* This intuitively evident property we assume. It is called the Archimedian property.

Now we are prepared to prove that the set of rational numbers is dense according to the second definition of density in Section 5.10, that is, the theorem written above. Let s and b be two real numbers. If they are of opposite sign, then 0 is a rational number between them and our result is shown. If we can prove the theorem for two positive real numbers, it will also be true for two negative real numbers, since we can find a rational number between the absolute values of the given real numbers and use its additive inverse. Hence we assume that s and b are positive and take $b > s$. Now there is a rational number t less than $b - s$, since by choosing $t = \frac{1}{n}$ and making n sufficiently large, we can make t as small as we please. It is intuitively evident from the geometric picture that if t is less than $b - s$, some integral multiple of t must be between s and b. To give a more precise proof we use the lemma above and find an integer q and a real number r such that

$$s = qt + r\,, \quad 0 \leq r < t\,.$$

First,

$$(q + 1)t = qt + t = s - r + t > s\,,$$

since $t > r$. Also

$$(q + 1)t = s + t - r < s + (b - s) - r = b - r \leq b.$$

Thus we have shown that the rational number $(q + 1)t$ is between s and b, and our proof is complete.

Exercises

1. Solve the following inequalities:

   a) $2x + 3 < 5$ b) $-2x + 5 > 7x - 2$

   c) $2 > \frac{1}{x - 3}$ d) $\frac{2}{x - 5} < \frac{7}{x + 6}$

2. Suppose in the lemma of this section $s = \pi$ and $t = 2$. Then find q and r.

3. Indicate on the number line the sets of points which satisfy the following conditions:

   a) $x > 1$ and $x \leq 4$ b) $x < -1$ or $x > 4$

   c) $x > 1$ or $x \leq 4$ d) $x < 1$ and $x > 4$

   e) $|x| < 3$ f) $|x| \geq 3$

4. In the following the letter a stands for one of $<$, $\leq$, $>$, $\geq$, and b one of the same set, while c stands for "or" or "and." Consider the statement:

$$(x \text{ a } 1) \text{ c } (x \text{ b } 4) .$$

(For instance, if a is $<$, b is $>$ and c is "or." it becomes $x < 1$ or $x > 4$.) Put in the proper sign or word for a, b, and c where possible so that the set of points x on the number line is:

a) a line segment
b) a ray
c) the whole line
d) a ray without the end-point
e) the empty set
f) two rays
g) two line segments .

Where your answer is "impossible," explain why.

Problems

1. Recall that in Section 4.9 we defined a set D of numbers of the form $\frac{n}{2^k}$ where n is an integer and k is a whole number. Show that if a and b are any members of D, then there are other numbers q and r of D such that

$$a = bq + r, \; 0 \leq r < b .$$

2. Recalling the second definition of density in Section 5.10, prove that between any two real numbers there is a number of the set D defined above and hence, by this definition of density, the set D is dense.

3. Let a , b , and c be three positive numbers of the set D defined above. Prove that there are numbers q and r of D such that

$$a = bq + r, \; 0 \leq r < c .$$

One way to show this is to use the result of the previous problem.

9.4 Why solve equations or inequalities?

Traditionally algebra was sometimes taught as if the whole purpose of the subject were to solve equations. And then when the grand climax appeared, the equations were of the puzzle type. Most of us enjoy solving puzzles, and they are edifying, but a mere puzzle should not be the aim of a year's study. Most of the equations which one would want to solve in everyday life by algebra require much more complex processes than can be discussed in a high school class, and in many cases the use of a machine is better. Certainly there are not many really practical problems which require the solution of a linear equation. Here is one.

Problem. A theater owner does not want his cashier to have to be bothered with anything less than half dollars. He wants a net gain, not counting expenses, for instance, of about $6.00 for the best seats in the house. But there is an entertainment tax of 10 per cent. What should he set the overall price to be?

He notices that 10 per cent of $6.00 is 60 cents. So if he charged $6.50 he would be a little short and $7.00 a little ahead. He asks himself, what would be my net gain for these prices? Now if N is the net gain, the price charged would be $N + \frac{N}{10} = N(\frac{11}{10})$. So he has two equations to solve:

$$N\left(\frac{11}{10}\right) = 6.50 \text{ , that is, } N = (6.5) \cdot \frac{10}{11} = \frac{65}{11}$$

$$N\left(\frac{11}{10}\right) = 7.00 \text{ , that is, } N = 7 \cdot \frac{10}{11} = \frac{70}{11}$$

This would give him the general result that his net gain would be $\frac{10}{11}$ of the price charged. In these two cases $N = \$5.91$, and $6.36 would be the net gains. Thus he has to decide between these two prices. Doing it this way instead of by trial has the advantage that he has a ratio $\frac{10}{11}$ which he can use for determining his net gain on the other priced seats as well.

This type of problem has many variations. It could be used in determining the wholesale price of an article if you know the retail price and retail mark-up. In these days of government contracts, if you know the percentage of the overhead and the total amount, you can find what there is to spend before overhead.

What should be emphasized a little more is the role of algebra in simplifying calculations. For instance, suppose there are three cities--A, B, and C--with B between A and C. You know the distances and want to find how far from A is the midpoint between B and C. The most natural way to do it is to add half the distance from B to C to the distance from A to B. But a slightly shorter way is to find half the sum of the distances of B and C from A. That the two are equivalent is shown by the following, where b is the distance of B from A and c the distance of C from A.

$$b + \frac{c - b}{2} = b + \frac{c}{2} - \frac{b}{2} = \frac{b}{2} + \frac{c}{2} = \frac{b + c}{2}$$

Actually it must be admitted that the average citizen gets along quite well without solving equations. To be sure, it is very common to use letters to stand for something--our whole language is this--but we do not add them unless we wish to be facetious as in

USA + FDR = NEW DEAL .

At least if we subtracted FDR from both sides, the equation would not be true, nor in fact would it make any sense.

Moreover, for the person whose most complicated arithmetical transaction is making out his income tax, algebra is dispensable. But if he wishes to be informed about what is going on in the world around him, he certainly needs to know the properties of numbers which pervade our lives. And these are algebraic properties. See Section 9.8.

9.5 Relations and functions

So far we have considered only the solutions of equations involving one letter, though we have considered many identities involving two or three letters like $ab = ba$ or $a + (b + c) = (a + b) + c$. Consider the simple equation:

$$x + 3 = y .$$

Here we can choose x as we please, and the equation will be true if y is 3 more than the x we chose. This expresses a relationship between two numbers x and y. Two other ways in which this same relationship could be expressed are:

$$y - x = 3 \quad \text{and} \quad x = y - 3 .$$

The second of these shows that we could also choose y as we please if we allow x to be any integer or rational number, but that if x has to be a positive rational number, the only values of y which yield a value of x are those greater than 3.

Another relationship would be expressed by the inequality:

$$x + 3 < y$$

In this case again we can choose x as we please and to satisfy the inequality, y must be greater than $3 + x$.

In both cases above we have a pair of numbers (x,y) and a relationship between them. We would even include the relationship implied by the equation $xy = 0$. For this equation if $x = 0$, we can replace y by any number and have the equation true. Also if $y = 0$, we can replace x by any number and have the equation true. But if $y \neq 0$, there is only one x, $x = 0$, for which the equation holds.

There are a number of advantages in representing these relationships graphically. Here the reader should recall Section 5.11 where the ordered pairs of numbers (a, b) correspond to points in the plane.

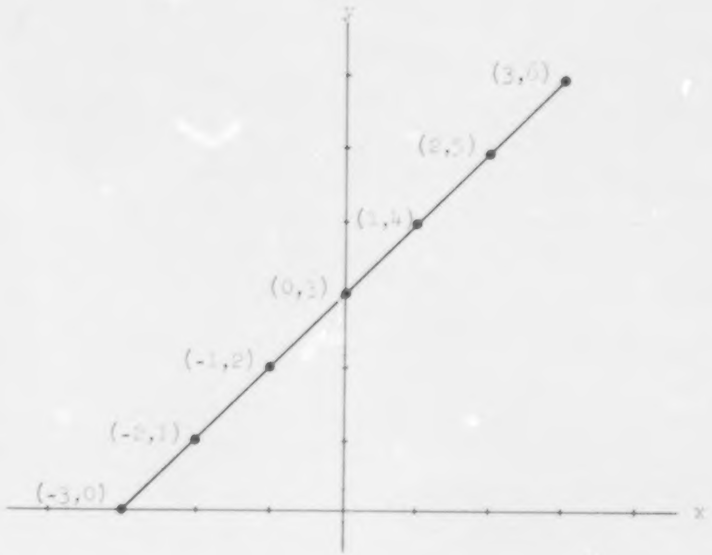
Consider the first equation of this section

$$x + 3 = y .$$

We could make a table of some of the pairs of values for which this equation is true:

x	-3	-2	-1	0	1	2	3
y	0	1	2	3	4	5	6

If we represent these points on graph paper, we would have a picture like this:



Of course these are not the only points. Another, for instance, would be $(3\frac{1}{2}, 6\frac{1}{2})$ or, indeed, $(\sqrt{2}, \sqrt{2} + 3)$.

It looks as if these points all lie on a straight line. In fact, it is true that the solution set of the equation consists of the pairs of coordinates of the points of the straight line connecting, for instance, points $(-3, 0)$ and $(0, 3)$. This is proved in Appendix IV for those interested.

In fact, it is true that every linear equation (that is, $ax + by + c = 0$ for numbers a, b, and c) has a line as its graph, and every line is the graph associated in this manner with a linear equation.

The line representing $x = y$ is parallel to that for $x + 3 = y$, since the two equations have no common solution.

The inequality

$$x + 3 < y$$

is true when y is a number which is greater than $3 + x$. In terms of the graph, for any x the numbers satisfying the inequality will be the second coordinates of those points above the line. So the graph of the inequality is all those points above the line $x + 3 = y$. Similarly the graph of $x + 3 > y$ would be all the points below the line.

Notice that the graph of the equation and the inequality differ in one important respect: for the former there is just one y for each x and for the latter there are infinitely many. The former we call a function. Formally:

A function is a set, S, of ordered number pairs (x, y) such that no two distinct pairs of S have the same second element y. In this case we would say that "x determines y." This is not quite accurate, for there may be values of x in a function which have no corresponding y.

For instance:

$$\frac{1}{x - 3} = y.$$

For each x except $x = 3$ there is a value of y and exactly one. But for $x = 3$ there is no number y for which the equation is true. We still call this a function. But the inequality $x + 3 < y$ is not a function because y has an infinite number of values for each x.

Sometimes a given equation works both ways; that is, each of x, y is a function of the other. We have the following pairs of equivalent equations:

$$x + 3 = y \text{ and } x = y - 3$$

$$\frac{1}{(x - 3)} = y \text{ and } x = 3 + \frac{1}{y} .$$

In these cases we say that the function has an inverse.

But not all functions have inverses. Consider the equation

$$y = x^2 .$$

It is a function since there is only one y for each x. But for a given non-negative y there are, except for $y = 0$, two possible values of x : $\sqrt{y}$ and $-\sqrt{y}$. However, even here if x is restricted to be a non-negative real number, for instance, and y to be non-negative, the inverse function exists, for then for each value of y there is exactly one value for x.

Graphically, we know that there is a one-to-one correspondence between ordered pairs of real numbers (x,y) and the points in the plane. (Recall that, starting with a pair of axes, x represents the horizontal coordinate and y the vertical one.) So graphically such a function may be represented by a set of points with the property that no two of the set are on the same vertical line. The function will have an inverse if no two points of the set are on the same horizontal line, since the roles of x and y are reversed. Figures 1a and 2a represent functions with inverses and 1b and 2b functions without inverses.

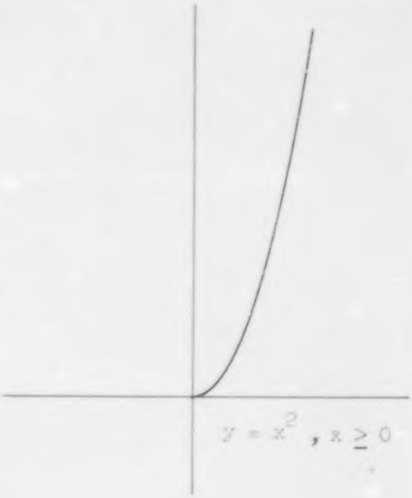


Figure 1a

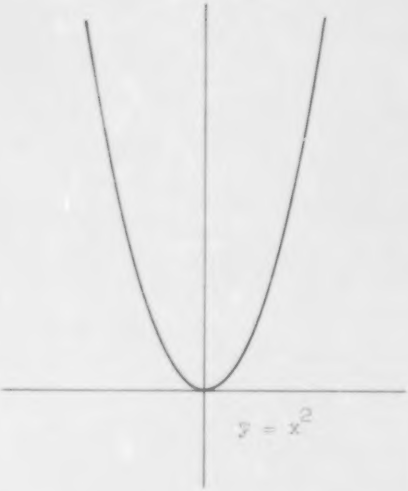


Figure 1b

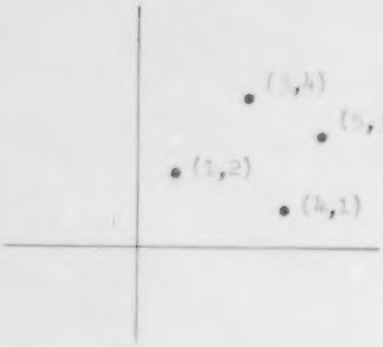


Figure 2a

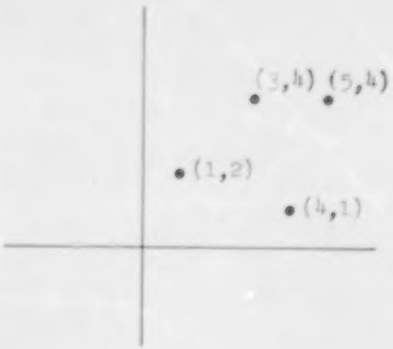


Figure 2b

Of course there are relationships and functions which are not given in terms of equations. Figures 2 above are two examples. The following table is another example of a function. This gives the heights of ten persons. It is a function since every person has a unique height.

Person number	1	2	3	4	5	6	7	8	9	10
Height in inches	48	73	62	53	76	69	71	52	48	66

Exercises

1. In Theorem 4 of Section 4.4 it was proved that if 1 is the g.c.f. of a and b, then the equation

$$ax + by = 1$$

has solutions in integers x and y. What does this tell us about the line $3x + 4y = 1$? What does this tell us about the general line $ax + by = 1$ when 1 is the g.c.f. of a and b?

2. Carrying on the ideas of the previous exercise and noting that

$$6x + 15y = 1$$

has no solutions in integers x and y, what does this mean in terms of the graph of this line?

3. An housewife spent $1.00 on oranges and bananas. If oranges cost 4 cents apiece and bananas 3, draw a graph indicating her possible purchases.

9.6 Applications to ordered pairs

Here the reader should recall Section 7.7 in which there was set up a correspondence between negative integers and ordered pairs of whole numbers. There the number $a - b$ was made to correspond to the ordered pair (a,b). There we found that all the points defined by ordered pairs corresponding to a given number were lattice points on a line parallel to the line through the points $(0,0)$ and $(1,1)$. More specifically, all ordered pairs of whole numbers corresponding to the number 1 are coordinates of lattice points on the line $x - y = 1$; all pairs corresponding to -2 are coordinates of lattice points on the line $x - y = -2$, etc.

We could set up another kind of correspondence as follows:

$$a - 3b \text{ corresponds to the ordered pair } (a,b).$$

Then the ordered pairs corresponding to the number 1 would be coordinates of the lattice points satisfying the equation $x - 3y = 1$; all pairs corresponding to -2 would be coordinates of lattice points satisfying the equation $x - 3y = -2$, etc.

One can also, in accordance with this correspondence, set up definitions of addition and multiplication of number pairs. Specifically

$$(a - 3b) + (c - 3d) = (a + c) - 3(b + d) \text{ corresponds to } (a,b) + (c,d) = (a + c, b + d)$$
$$(a - 3b)(c - 3d) = (ac + 9bd) - 3(bc + ad) \text{ corresponds to } (a,b)(c,d) = (ac + 9bd, bc + ad)$$

We could proceed further with this correspondence and add and multiply lines as in Section 7.11 but it seems scarcely worth while here.

In Section 7.11 we considered also the positive rational numbers as ordered pairs of whole numbers. In this case the correspondence is between the number $\frac{a}{b}$ and the pair (a,b). Then

$$(x,y) = (a,b) \text{ if and only if } xb = ya,$$

which is the equation of a line through the origin. Recall that the number pairs (a,b) and (c,d) will correspond to the same rational number if and only if $ad = bc$. This is just the condition that the point (c,d) lie on the line $xb = ya$, since if we replace x by c and y by d, we have $cb = da$. Thus the point (c,d) will be on the line $xb = ya$ if and only if the number pairs (a,b) and (c,d) correspond to the same rational number. In other words, all the points corresponding to a given rational number are lattice points on a single line through the origin.

9.7 Pairs of linear equations

We have found that a single linear equation in two letters or unknowns usually has an infinite number of solutions. But if we have a pair of linear equations and wish a common solution what may happen can be seen by recalling that a linear equation represents a line. So one of three things may happen:

1. The lines are the same.
2. The lines intersect in only one point.
3. The lines do not intersect, that is (since they are in the same plane) they are parallel.

Each of these categories has its equivalent statement, of course, in terms of the equations, as follows:

1. The equations represent the same line, in other words they are equivalent equations. For example: $2x + 3y = 5$ and $4x + 6y = 10$.
2. There is just one common solution, that is, the intersection of the two solution sets is one number pair. For example $2x + 3y = 5$ and $2x - 3y = -1$ with the solution (1,1).
3. There is no common solution, that is, the intersection of the two solution sets is the null set. For example, $2x + 3y = 5$ and $2x + 3y = 7$.

In the last case it is easy to see that there is no solution in common because $2x + 3y$ cannot be simultaneously equal to 5 and 7.

Here, as for a single linear equation, the process of solution is to devise, by certain procedures, a sequence of equivalent pairs of equations until in the end the pair is, in the case of a single solution, something of the form $x = m$, $y = n$.

We first illustrate the process by a numerical example and then show why the sequence is a set of equivalent pairs.

$$2x + 3y = 5$$
$$3x - 2y = -12.$$

Here if we can manage it so that we have an equivalent pair of equations in which the coefficients of x are the same, then we can subtract and get an equation in y alone. If we multiply the first equation by 3 and the second by 2 this will be accomplished. So we have the pair

$$6x + 9y = 15$$
$$6x - 4y = -24.$$

This pair has the same solution set as the first because this is true of each equation of the sequence. Then if we subtract the second from the first we get

$$13 \cdot y = 39, \text{ or, } y = 3.$$

In the first of the given equations, $y = 3$ implies

$$2x + 9 = 5, \text{ or } x = -2.$$

There are at least two ways to look at the above process. The first is this. We suppose that there is an ordered pair (x,y) which satisfies both equations. If so, this same pair will satisfy the second pair of equations. Thus for this pair

$$6x + 9y - 15 = 0$$

and $$6x - 4y + 24 = 0.$$

So if a pair of numbers satisfies both equations above, it certainly satisfies

$$6x + 9y - 15 - (6x - 4y + 24) = 0$$

that is,

$$13y - 39 = 0.$$

This tells us that if there is a common solution, y must be 3 and by substitution we find that x must be -2. Thus we have shown that if there is a solution it must be $(-2,3)$. Then by substituting this pair in the given pair of equations we verify that indeed it is a solution.

The second point of view is this. We have the given pair of equations which is equivalent to the second pair for the reasons pointed out above. Our third pair is

$$6x + 9y - 15 - (6x - 4y + 24) = 0, \text{ that is, } 13y - 39 = 0$$

and $$2x + 3y = 5.$$

Just why this pair is equivalent to the previous pair can be seen a little more easily probably in terms of the general discussion below.

Then the pair

$$y = 3$$

$$2x + 3y = 5$$

is equivalent to $$y = 3$$

and $$2x = -4$$

or finally, to the pair $y = 3$, $x = -2$. By this line of reasoning we know that if we have not made a mistake in the process, $(-2,3)$ is the solution and there are no more.

It is perhaps a little easier to see what is going on if we use letters. Suppose the two given equations are

$$f = 0$$

$$g = 0.$$

(In our illustration $f = 2x + 3y - 5$ and $g = 3x - 2y + 12$). There are two fundamental operations.

1. Multiply an equation by a non-zero constant. Thus $f = 0$ and $af = 0$ for $a \neq 0$ are equivalent equations and

   $$af = 0,\ a \neq 0$$
   $$bg = 0,\ b \neq 0$$

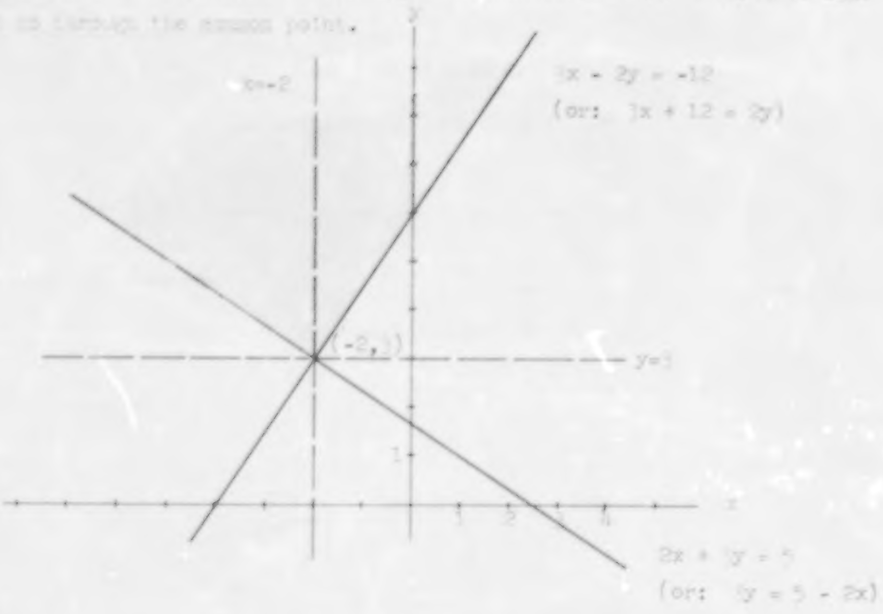
   is an equivalent pair.

2. Subtract one equation from another: and have a pair

   $$af - bg = 0$$
   $$af = 0$$

This is an equivalent pair because if $af - bg = 0$ and $af = 0$, then $bg = 0$. Conversely if $af = 0$ and $bg = 0$, then $af - bg = 0$. We could of course also have used the pair $af - bg = 0$ and $bg = 0$.

Notice that in all cases we must continue with a pair. Geometrically this has a rather striking interpretation. The first process above does not change the lines, the second process replaces one of the lines by one which goes through the same common point. Thus from a geometrical point of view the solution is the process of finding a horizontal line and a vertical line which go through the common point.



This same process works for inequalities as well. Suppose we have

$$2x + 3y > 5$$

$$3x - 2y < -12.$$

The first inequality is equivalent to the inequality

$$3y > 5 - 2x.$$

Now on the line $3y = 5 - 2x$, $3y$ and $5 - 2x$ are equal. Thus for the inequality $3y > 5 - 2x$, $3y$ and thus y is greater than the point on the line corresponding to the same x. This means geometrically that the points which satisfy the inequality $3y > 5 - 2x$ lie above the line $3y = 5 - 2x$.

The second inequality is equivalent to $3x + 12 < 2y$. Hence the points which satisfy it are the points above the line $3x - 2y = -12$. Thus those points which satisfy both are those which are both above the first line and above the second as in the figure.

Some may prefer to test an inequality by checking it for one point. A convenient point to use for this purpose is $x = 0$, $y = 0$. In the case of $2x + 3y > 5$, we see that $x = 0$, $y = 0$ does not satisfy the inequality. Since, from the graph, the point O is in the half of the plane below the line $2x - 3y = 5$, the part of the plane which satisfies the inequality is that above the line. Looking at the other inequality, $3x - 2y < -12$, we see that the coordinates of the origin do not satisfy this inequality either. Since the origin lies below the line $3x - 2y = -12$, the points which satisfy the inequality lie above the line.

Exercises

1. Find all the solutions of each of the following pairs of equations algebraically and draw their graphs:

a)	$x + y = 1$	and	$x - y = 3$
b)	$2x + 3y = 7$	and	$3x - 2y = 4$
c)	$2x + 3y = 7$	and	$4x + 6y = 14$
d)	$2x + 3y = 7$	and	$4x + 6y = 15$

2. For each pair in the previous exercise, replace the first equality by $>$ and the second by $<$. Then show graphically the points which satisfy each pair of inequalities.

3. Recall that $x^2 - y^2 = (x - y)(x + y)$ and that if x and y are integers such that

$$x^2 - y^2 = 9,$$

then

$$(x - y)(x + y) = 9,$$

that is, $(x - y)$ and $(x + y)$ are two integers whose product is 9. Thus find all the pairs of integers x and y which satisfy the equation

$$x^2 - y^2 = 9.$$

4. Find all the solutions in integers of each of the following equations:

   a) $x^2 - y^2 = 1$
   b) $x^2 - y^2 = 40$
   c) $x^2 - y^2 = 14$

5. Guess from the examples in the previous exercise the integral values of c for which the following equation has a solution in integers x and y.

$$x^2 - y^2 = c.$$

Then see if you can substantiate your guess.

Problems

1. Suppose $f = ax + by + c$ and $f' = a'x + b'y + c'$. Show that $f = 0$, $f' = 0$ have a single common solution if and only if

$$ab' - a'b \neq 0.$$

2. In the previous problem, what two kinds of things may happen if the equality $ab' - a'b = 0$ holds?

3. Using the notation of Problem 1, for what values of r, s, t, u will the given pair be equivalent to the pair:

$$rf + sg = 0$$
$$tf + ug = 0?$$

9.8. Applications

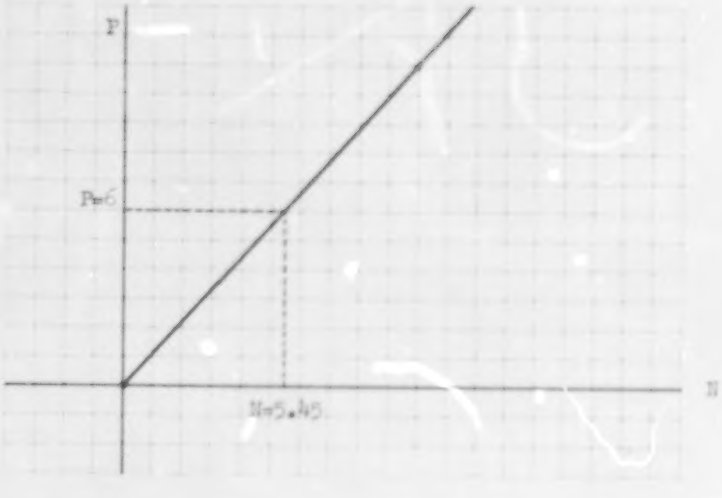
Not only are graphs useful in visualizing what is going on in the above algebraic processes but they can also be useful if one wants approximate solutions. First consider the problem of the theater owner in Section 9.4. Here there is a formula:

$$N = (\frac{10}{11})P$$

where P is the price including tax and N is the net return to the owner. A simpler way to express this would be:

$$11N = 10P.$$

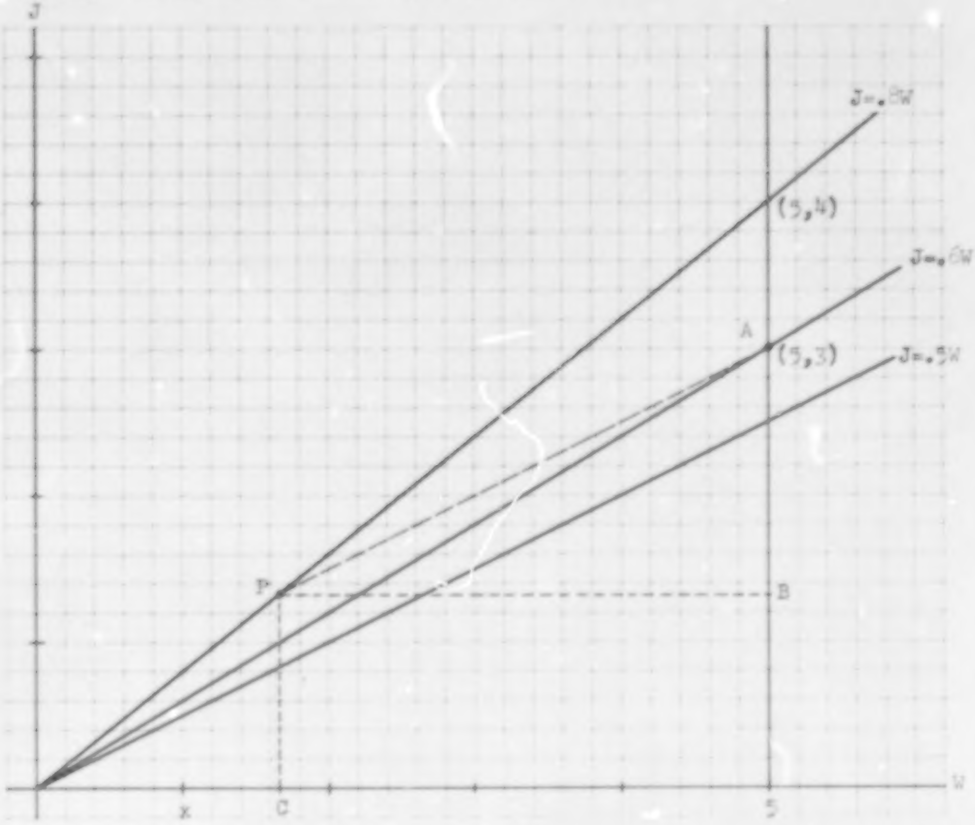
Making use of our knowledge that the graph is a straight line, we notice that it is satisfied by the pairs N = 0 = P and N = 10, P = 11 that is, it goes through the points (0,0) and (10,11). Thus on the graph paper it is an easy matter to draw the line as in the figure. This is a function which has an inverse, that is, if you know P you can find N and vice-versa. So to find the net return for a price of $6 you find the point of the line which corresponds to P = 6 and see what the value of N is at that point.



The graphical technique is also useful in mixture problems. Suppose one has two mixtures of coffee, the first of which is 80% Java and 20% Mocha, while the second is 50% Java and 50% Mocha. The object is to use proportions of these to get a five pound bag which is 60% Java and 40% Mocha. If W stands for the weight in pounds of the mixture, and J the number of pounds of Java, we have in the three cases:

$$J = .8W, \qquad J = .5W, \qquad J = .6W.$$

The graphs of these are three lines through the origin. The technique is as follows. (We leave the reasons for a problem.)



Plot the three lines, measuring weight along the horizontal axis and J along the vertical axis. For the third line find the point where W = 5, that is, the point (5,3). Then draw the line parallel to the second line through this point (5,3). Where it cuts the first line will be a point, P, whose first coordinate, x, is the weight of the first mixture which should be used.

Of course this problem can be solved algebraically. Though this method is quite different and probably not as interesting, it does provide a more accurate solution. This also is left as a problem.

Our third example involves inequalities. A certain dealer has a warehouse which he wishes to use to store bags of grain and cement. A bag of grain takes twice the space of a cement bag but weighs 75 pounds instead of 50. The capacity of the warehouse is A bags of grain. Since the floors are weak, he cannot store more than B pounds. On the other hand, grain is worth three times as much per bag as the cement. He wants to arrange his storage so as not to exceed the capacity of the warehouse either in space or weight but to maximize the value of what he stores. What should be the distribution? To solve this, or show how it would be solved, let g be the number of bags of grain and c the number of bags of cement. Then for the space requirements

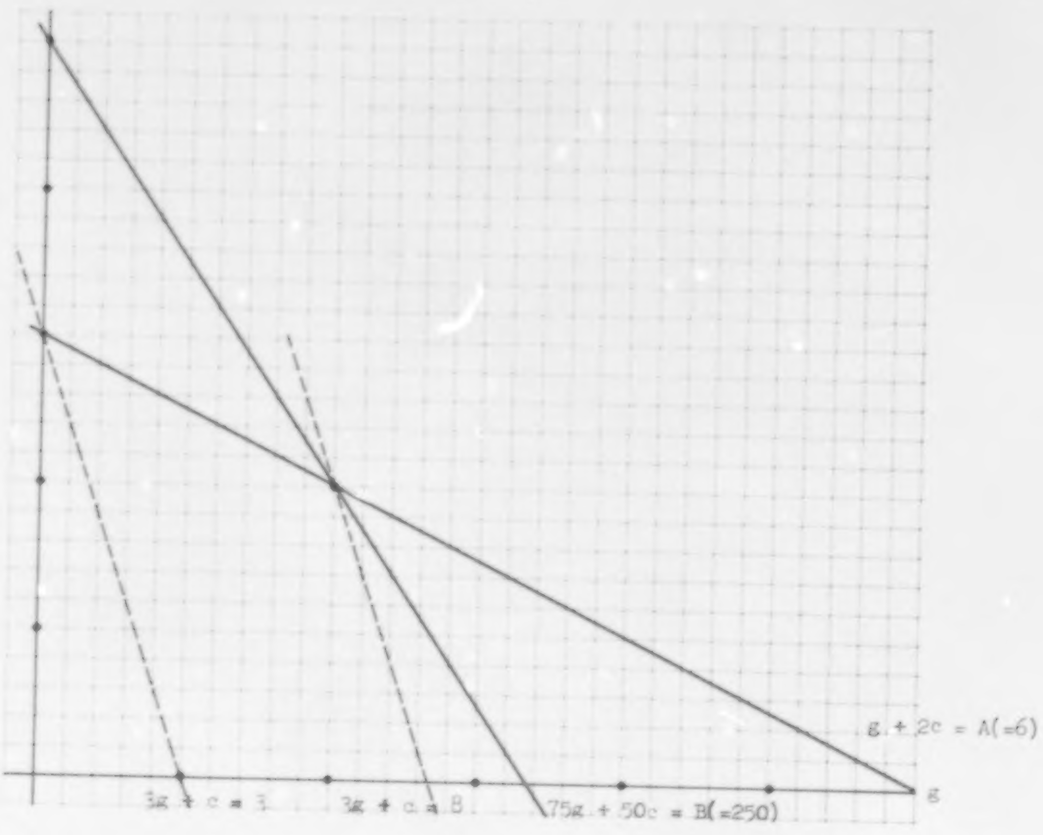
$$g + 2c \leq A.$$

For the weight requirements

$$75g + 50c \leq B.$$

If he wants to maximize the value, he must maximize

$$3g + c.$$



The graph of the first inequality is the set of points below and on the line $g + 2c = A$ and the graph of the second is the set of points below and on the line $75g + 50c = B$, as shown in the figure, in which it is assumed that the common point of the two lines has both coordinates positive. Now what about $3g + c$? Consider the lines $3g + c = x$. These will be parallel lines since no two intersect. So it is quite obvious from the graph (at least for $A = 6$ and $B = 250$) that the maximum x will be $3g + c$ where (g,c) is the point of intersection of the first two lines. (For $A = 6$, $B = 250$, the point of intersection is $(2,2)$ and the value of $3g + c$ at this point is 8. It is realized that these values for A and B are not realistic in terms of the problem.)

Exercises

1. The radiator of a car is full of a mixture of anti-freeze and water, the percentage of water being 70%. The tank holds five gallons. How much of the tank must be drained and replaced by pure anti-freeze to get a 50% mixture? Solve this algebraically and graphically.

2. Jack Sprat could eat no fat and his wife could eat no lean. Jack's minimum daily requirement is 1.2 pounds of lean and his wife's 1. pound of fat. Now pork is 30% lean and 70% fat while beef is 60% lean and 40% fat; but beef costs $\frac{3}{2}$ as much as pork. How many pounds of each kind of meat should they buy to meet the minimum requirements and still be as economical as possible?

3. In the mixture problem of this section the solution was obtained by drawing a line through the point A parallel to the line $J = .5W$, and finding the first coordinate of its intersection with $J = .8W$. Could one also obtain the solution by drawing a line through the point A parallel to the line $J = .8W$, and finding the first coordinate of its intersection with $J = .5W$? Why or why not?

4. Find a solution of the mixture problem with .8 replaced by the letter r, .5 by the letter s and .6 by the letter t, where r, s and t are numbers between 0 and 1 and t is between r and s.

Problems

1. Give an algebraic solution of the problem above about the coffee mixtures.
2. Justify the method used for the geometrical solution of the coffee mixture problem.
3. For the example above involving a warehouse, find conditions on A and B that the point of intersection of the two lines have both coordinates positive. What are the possibilities if this condition is not satisfied?

References

1 (Chapter 10), 8 (Chapters 6 and 7).

Appendix 1

SETS

1. *Introduction*.

We have various names for collections of things, men or animals: herd, flock, squad, regiment, covey. The word "set" includes all of these, and more, without any connotation of the kind of thing considered. In fact, it could include a collection of thoughts, ideas, or what you will. The members of the set again could have various names but, to be noncommittal, in mathematics we call the members of the set its "elements." The chief thing important about a set is that for one or another reason any given entity is either an element of the set or not. All the houses on a given street constitute a set because either something is a house on the street or it is not. We may not know whether John's house is on that street but we know that it is either an element of houses on the street or it is not.

2. *Relationships Between Sets*.

Suppose A and B stand for two sets. There are a number of possible relationships between them. They may be equal; by this we mean that every element of A is in B and vice versa. If only the first part of this relationship holds, that is if every element of A is in B, we say that set A is *contained in* B or is a *subset* of B, or that B contains A. We write this relationship*

$$A \subset B .$$

---

[*] Often the notation $\subseteq$ is used in place of our $\subset$. When this is the case the symbol $\subset$ excludes equality, whereas for our usage $\subset$ includes equality.

As we have just written:

$$\text{If } A \subset B \text{ and } B \subset A \text{ then } A = B$$

and

$$\text{if } A = B, \text{ then } A \subset B \text{ and } B \subset A .$$

Thus if A is the set of even counting numbers: 2, 4, 6, ..., and B is the set of whole numbers: 0, 1, 2, 3, 4, 5, 6, ..., then A is a subset of B. If A is the set consisting of the numbers 6, 7, 8 and B of 7, 8, 6 then the two sets are equal, or the same.

3. New Sets from Old.

Sometimes two sets have elements in common. These elements form a set since if we know when something is in set A and when it is in set B, we know when it is in both. This set is called the intersection of the two sets and designate it by $A \quad B$.

Thus $A \cap B$ denotes the set of elements common to A and B. Suppose A is the set of countries in Central America: Guatemala, El Salvador, Honduras, Nicaragua and Costa Rica. Let B be the set of countries in the western hemisphere whose names begin with C: Canada, Costa Rica, Chile, Colombia. Then the intersection of these two sets is: Costa Rica.

It sometimes happens that two sets have no elements in common. For instance, let R be the set of countries in the western hemisphere whose names begin with U: United States of America, Uruguay. Since there are no countries in Central America whose names begin with U, there are no elements common to A and R. So that we can say that the intersection of any two sets is a set, and for other reasons, we define what we call the null or empty set and denote it by $\emptyset$. Thus we write

$$A \cap R = \emptyset .$$

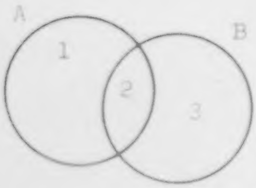
The null set is contained in every set. Why are all null sets the same? If the intersection of two sets is the null set, we call them disjoint.

We can form another set from two sets by considering all the elements which are in one or both. This is called the union of two sets. Thus the union of sets A and B above is:

Guatemala, El Salvador, Honduras, Nicaragua, Costa Rica, Canada, Chile, Colombia.

We designate the union of A and B by $A \cup B$.

These sets may be pictured as in the following figure:



Here the set A consists of the two regions numbered 1 and 2, while B consists of the regions numbered 2 and 3. The union of these two sets, $A \cup B$, consists of the regions numbered 1, 2 and 3. The intersection, $A \cap B$, is that region numbered 2.

Problems

1. If A and B are two lines in a plane, what possibilities are there for their intersection? A and B are sets of what?
2. Prove that all null sets are equal.
3. Show that the intersection of two sets is a subset of each.
4. If A and B are two sets, prove that they are both subsets of their union.
5. If $A \cap B = B$, what relation must exist between the two sets?
6. If $A \cup B = B$, what relation must exist between the two sets?
7. Show that $A \cup B = B \cup A$, $A \cap B = B \cap A$, $A \cup (B \cup C) = (A \cup B) \cup C$. Also show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4. The Relative Complement and Cartesian Product.

There are two other sets which can be obtained from a pair of given ones and which we have used in one form or another. The first is the relative complement, denoted by

$$A - B.$$

This stands for all the elements of A which are not in B. If B is a subset of A and A is a finite set, the number of elements in A - B is the number of elements in A minus the number in B. Thus the relative complement has a close connection with subtraction. Notice that

$$A - B = A$$

if $A \cap B = \emptyset$ and only in this case. It is also true that

$$(A - B) \cup B = A \cup B.$$

Another relationship is

$$(A - B) \cup (B - A) = A \cup B - A \cap B,$$

that is, the set of elements in A or B but not both.

The cartesian product of two sets A and B is the set of all ordered pairs (a,b) where a is an element of A and b an element of B. We have considered such a product when A and B are both the set of whole numbers--these are coordinates of lattice points. If the number zero is excluded from the set B, the nonnegative fractions form the elements of a cartesian product.

5. Evaluation.

In many situations the importance of sets these days is somewhat overrated. The language of sets is useful because it serves to focus ideas which are common property. From the author's point of view, it is out of place to try any detailed study of sets at the secondary school level or before. The language should be used where it is convenient and, where the properties of sets are like those of the numbers, they should be described and dealth with.

References

1 (Chap. 6), 8 (Chap. 1), 22 (Chap. 2).

Appendix 2

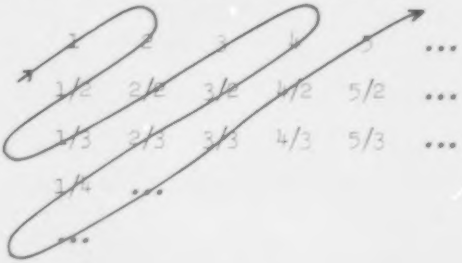
COUNTABLE SETS

The title of this appendix is in mathematics a technical term. If a set can be put into one-to-one correspondence with a subset (of course this includes the possibility of the set itself) of the set of natural numbers it is called a countable set (or denumerable). Thus the number of sands in the sea is a countable set though we cannot count them. The set of even numbers is countable since we can set up the one-to-one correspondence n ↔ 2n, that is, 1 corresponds to 2, 2 to 4, 3 to 6, ... , any integer to its double. The integers form a countable set for we can set up the following correspondence:

0	1	-1	2	-2	3	-3	4	-4 ...
1	2	3	4	5	6	7	8	9 ...

Thus each integer would be associated with a natural number.

In fact, the same can be done for ordered pairs where the numbers are, for instance, natural numbers. Here it is a little easier to write the ordered pairs as fractions and use the following scheme



where the curve indicates the order in which they are to be taken. This shows that the set of such fractions is countable. By discarding duplicates as they arise, we can also show that the positive rational numbers are countable.

In fact if S and T are two countable sets, we can show that the set of ordered pairs (s,t) where s is an element of S and t an element of T is ordered: first by associating the first numbers of the pairs with the natural numbers, then similarly with the second numbers of the pairs, third using the ordering of the pairs which we designated for the lattice points.

In a similar fashion it would be easy to show that the ordered triples (a,b,c) where a, b, c are natural numbers is a countable set.

With this discussion it is a little hard to imagine how one could find a set of numbers which is not countable. A most remarkable fact is that the set of real numbers is not countable. The proof is simple. It consists in assuming that we have the real numbers associated in one-to-one fashion with the natural numbers and then showing that we have not included them all. Suppose we had such an ordering of the real numbers in decimal form between zero and 1. Let the first four numbers be:

$$.a_1a_2a_3a_4a_5\ldots$$

$$.b_1b_2b_3b_4b_5\ldots$$

$$.c_1c_2c_3c_4c_5\ldots$$

$$.d_1d_2d_3d_4d_5\ldots$$

where the letters stand for the digits in the decimal representation. The listing would continue like this indefinitely. Now we show that there is a real number not in the list. It is

$$.abcde\ldots$$

where a, the first digit, is different from a_1 and 9; b, the second digit, is different from b_2 and 9; c, the third digit, is different from c_3 and 9; d, the fourth digit, is different from d_4 and 9, and so forth. (We avoid 9 so as not to have two decimal expansions representing the same real number. See Section 6.5.) This number

$$.abcd\ldots$$

is not equal to the first number in the above list, since its first digit is different; it is not equal to the second because its second digit is different, etc. Thus, whatever ordering one has, that is, whatever way the one-to-one correspondence is set up, we will have left out one number and hence not included them all.

We have thus shown that the set of real numbers is not countable. This shows that we cannot get the real numbers by considering pairs, triples, quadruples, or in fact n-tuples of natural numbers since all these "tuples" would form a countable set.

References

4 (Appendix A), 13 (Appendix C).

Appendix 3

COMPLEX NUMBERS

1. Introduction.

Looking back over our development of the number system you will recall that each enlargement was in response to a need. For the rational numbers we wanted to solve the equation $ax = 1$ when $a \neq 0$ and so we invented numbers $\frac{1}{a}$ which satisfied this equation. We needed to solve $a + x = 0$ and so we invented the negative numbers ^-a. We needed to solve $x^2 = 2$ and like equations and so we invented the real numbers. In each case we made the invention so that as many as possible of the previous properties would be preserved.

There are still equations left to conquer--that is equations with real coefficients which have no solutions in the set of real numbers. One such equation is $x^2 = -1$. We fill this gap by the invention of a new kind of number which is called a complex number. We begin by designating a number which is a solution of the equation $x^2 = -1$. We denote one such number by the letter i. And with this we add a whole set of numbers written

$$a + bi$$

where a and b are real numbers. Just as for fractions, we must define what we mean by equality of two such numbers, then their sum and finally their product. We accordingly give the following definitions:

1. $a + bi = c + di$ if and only if $a = c$ and $b = d$, that is, only if the two parts of each number are the same. Thus $2 + 3i = \frac{6}{3} + 3i$ because $2 = \frac{6}{3}$. But $2 + 3i \neq 3 + 2i$, since the corresponding parts are not equal. This implies $2 + 3i = x + yi$ for x and y real only if $x = 2$ and $y = 3$.

2. $(a + bi) + (c + di) = (a + c) + (b + d)i$. In other words, we add corresponding parts of the number to get the sum of two such numbers. One reason for defining the sum this way is that the commutative and distributive properties are preserved.

3. $(a + bi)(c + di) = ac + adi + bic + bdi^2 = ac - bd + (ad + bc)i$ because i is a root of $x^2 = -1$ and thus $i^2 = -1$. Also we want to have the usual properties of product and sum preserved.

In all of these definitions of course, all the letters except i are understood to stand for real numbers.

The set of all numbers with the above definitions is called the set of complex numbers. A number $a + bi$ where $b \neq 0$ is called an imaginary number. Thus the complex numbers are divided into two categories: the real numbers ($a + bi$ with $b = 0$) and the imaginary numbers ($a + bi$ with $b \neq 0$). In fact, in the complex number $a + bi$, the number a is often called the real part and bi the imaginary part, or, more strictly speaking, the pure imaginary part.

[illegible]. The Properties of the Complex Numbers.

We shall not here take the time or space to develop systematically the set of complex numbers. If we were to do this we would have to show that this set has all the properties of a field defined in Chapter VII (Section 7.[illegible]). We shall only show that division is possible except by zero in the set of complex numbers. To do this we want to find real numbers x and y such that

$$(a + bi)(x + iy) = 1.$$

This means that $ax + bix + aiy + byi^2 = 1$, that is,

$$(ax - by) + (bx + ay)i = 1 + 0 \cdot i.$$

Thus, by the definition of the equality of two complex numbers we have two equations to solve.

$$ax - by = 1$$
$$bx + ay = 0.$$

Using the technique developed in Section 9.7 we multiply the first equation by b and the second by a to get

$$bax - b^2y = b$$
$$abx + a^2y = 0.$$

If we subtract the upper from the lower we have

$$a^2y + b^2y = -b,$$
$$(a^2 + b^2)y = -b.$$

Thus if $a^2 + b^2 \neq 0$, we can solve the last equation for y and have

$$y = \frac{-b}{(a^2 + b^2)} . \tag{1}$$

To find the value of x we may substitute this value of y in $bx + ay = 0$ or start again with the given equations, multiply the first by a, the second by b and add. In either case we will have

$$x = \frac{a}{(a^2 + b^2)} . \tag{2}$$

Thus we have shown that if $a^2 + b^2 \neq 0$, the equations (1) and (2) give values of x and y which satisfy the original equation:

$$(a + bi)(x + iy) = 1.$$

It remains to show that $a^2 + b^2 = 0$ if and only if a and b are both zero, that is, $a + bi = 0$. Suppose $a^2 + b^2$ were zero with $b \neq 0$. Then we would have the following series of equalities:

$$a^2 = -b^2, \ (\frac{a^2}{b^2}) = -1, \ (\frac{a}{b})^2 = -1.$$

This would imply that the square of a real number, $\frac{a}{b}$, is equal to -1. This is impossible. Hence $a^2 + b^2 = 0$ with a and b real implies $b = 0$ and hence $a = 0$. So now we have shown that unless $a + bi = 0$, the equation: $(a + bi)(x + iy) = 1$ has a single solution in real numbers x and y.

This process is one which we would not like to have to repeat for many divisions. To find a shorter way, let us write down in another form, the number x + yi which we just found. It is

$$x + yi = \frac{a - bi}{a^2 + b^2}.$$

Thus

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2}.$$

Here we notice that we could have obtained the right side from the left by multiplying both numerator and denominator by a - bi since

$$(a + bi)(a - bi) = a + bai - abi - b^2i^2 = a^2 + b^2.$$

There is one difficulty in connection with the complex numbers which should be discussed. It stems from the fact that not only is i^2 equal to -1 but $(-i)^2$ as well, for $(-i)^2 = (-1)^2 i^2 = (-1)$. Thus there is really no way to distinguish between i and -i. We only know that the equation $x^2 = -1$ has two roots, i and -i. Then the question rises, what is $\sqrt{-3}$, for instance? From the notation it must be a root of $x^2 = -3$, but which root? To avoid ambiguity we make the agreement that, in terms of our chosen i,

$$\sqrt{-3} = i\sqrt{3}.$$

This choice gives us consistent results because then

$$(\sqrt{-3})(\sqrt{-3}) = (i\sqrt{3})(i\sqrt{3}) = i^2(\sqrt{3})^2 = -3,$$

which is what it should be. Also

$$(\sqrt{-3})(\sqrt{-2}) = (i\sqrt{3})(i\sqrt{2}) = i^2(\sqrt{6}) = -\sqrt{6}.$$

Note that for complex numbers

$$\sqrt{a} \cdot \sqrt{5} \neq \sqrt{ab};\ \text{e.g.,} \sqrt{-3} \cdot \sqrt{-2} \neq \sqrt{(-3)(-2)}.$$

A fundamental fact about the set of complex numbers is that all the roots of polynomial equations with complex coefficients are complex numbers. That is, not only are equations like $x^2 = -2$ solvable in complex numbers but every equation of the following form as well:

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ \dots \ + a_1 x + a_0 = 0$$

where the a's are complex numbers. This is much too difficult to prove here. Because of this property the set of complex numbers is often called an algebraically complete number system.

3. Complex Numbers as Number Pairs.

You may recall that in previous work we noted that a rational number could be considered as an ordered pair of integers (a,b) which corresponded to the fraction $\frac{a}{b}$. Similarly for negative numbers the number a - b could have been made to correspond to the number pair (a,b). Each of these would have its own definitions of equality, addition and multiplication. So, with a sidelong look at the previous section, we have for complex numbers the following corresponding definitions for the number pairs:

	a + bi form	number pair
Equality:	$a + bi = c + di$ if and only if $a = c$ and $b = d$	$(a,b) = (c,d)$ if and only if $a = c$ and $b = d$
Addition:	$(a + bi) + (c + di)$ $= (a + c) + (b + d)i$	$(a,b) + (c,d) = (a + c\ ,\ b + d)$
Multiplication:	$(a + bi)(c + di)$ $= ac - bd + (bc + ad)i$	$(a,b) \cdot (c,d) = (ac - bd\ ,\ bc + ad)$

These definitions take on special significance when we look at them from a graphical point of view. Here instead of the x and y axes, we have a horizontal axis which we call the axis of reals and the vertical axis which we call the *axis of pure imaginaries*. Then the point (a,b) corresponds to the complex number $a + bi$. Every point thus has a single complex number as its coordinate. Two points are the same if and only if the corresponding complex numbers are equal.

Next, what is the geometrical significance of the sum of two complex numbers? Here the reader should refer back to Section 7.7 where the sum of two number pairs was defined in the same manner as here for complex numbers. (Though equality and the product were differently defined, this makes no difference to the geometrical meaning of the sum). In that section and the solution of one of the problems, it was shown that the geometrical meaning of the sum of the numbers $a + bi$ and $c + di$ is this: Associate with each of these numbers the vector from the origin to the number. Then the vector which is the resultant of the two other vectors is that from the origin to the sum: $a + c + (b + d)i$. In other words, if O is the point 0, A the point $a + bi$ and B the point $c + di$, then the point $a + c + (b + d)i$ is the fourth vertex of the parallelogram two of whose sides are $\overline{OA}$ and $\overline{OB}$, as shown in the answer to problem 2 of Section 7.7. This is the reason for one use of complex numbers in physics.

The product of two complex numbers also has a geometrical significance which we partially establish in the two theorems below. Consider the complex number $a + bi$, the coordinate of a point A. We call the distance OA the *absolute value* of the number $a + bi$. Using the same notation we used for real numbers, we have:

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Also associated with $a + bi$ is an angle. Let T be the point with coordinate 1, then the angle TOA measured in a counterclockwise direction is called the *amplitude* of the number $a + bi$. Here we may write

$$\text{amp}(a + bi) = \angle TOA.$$

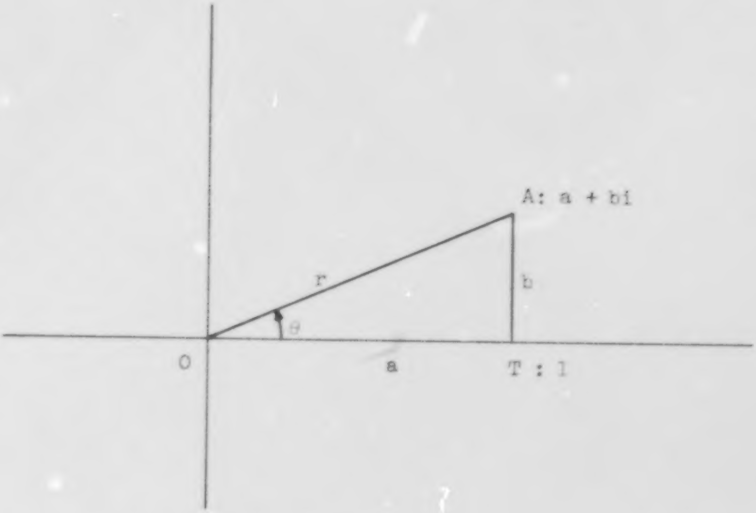


Figure 1

Notice that a complex number is determined by its absolute value and amplitude. (If $b = 0$, the angle TOA is the zero angle and has zero measure.) We now state two theorems:

Theorem 1. The absolute value of the product of two complex numbers is equal to the product of their absolute values, that is, if

$$(a + bi)(c + di) = r + si,$$

then

$$|a + bi||c + di| = |r + si|$$

that is

$$\sqrt{a^2 + b^2}\ \sqrt{c^2 + d^2} = \sqrt{r^2 + s^2}.$$

Theorem 2. The measure (in degrees) of the amplitude of the product of two complex numbers is equal to the sum of the measures of the amplitudes of the separate numbers or 360 less than this sum if the sum is greater than 360.

To prove Theorem 1 notice that the conclusion is equivalent to

$$r^2 + s^2 = (a^2 + b^2)(c^2 + d^2).$$

Now $(a + bi)(c + di) = ac - bd + (ad + bc)i$ and thus

$$r = ac - bd \text{ and } s = ad + bc.$$

Hence we wish to prove:

$$(ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2).$$

The left side is equal to:

$$a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2 =$$

$$a^2(c^2 + d^2) + b^2(c^2 + d^2) = (a^2 + b^2)(c^2 + d^2)$$

and our proof is complete.

The second theorem we prove only for the case when a, b, c, d are positive numbers. First we need an auxiliary result which is not subject to these restrictions.

Lemma. Let points A, B, C have coordinates α, β, and $\alpha + \beta$ where α and β are complex numbers; let points A', B', C' have coordinates $\mu\alpha$, $\mu\beta$, and $\mu(\alpha + \beta)$ where $\mu \neq 0$ and is a complex number. Assume that A, B and O, the origin, are distinct. Then the angles

$$\angle AOB \text{ and } \angle A'OB'$$

are congruent.

Proof: $$OA = |\alpha| , \ OB = |\beta| , \ OC = |\alpha + \beta|$$

$$OA' = |\mu\alpha| = |\mu||\alpha| , OB' = |\mu\beta| = |\mu||\beta| , \ OC' = |\mu\alpha + \mu\beta| = |\mu||\alpha + \beta|$$

by Theorem 1. Furthermore OA = BC and OA' = B'C' (see the figure). Thus if we consider the triangles OBC and OB'C' we have

$$|\mu|(OB) = OB' , \ |\mu|(OC) = OC' , \ |\mu|(BC) = B'C'$$

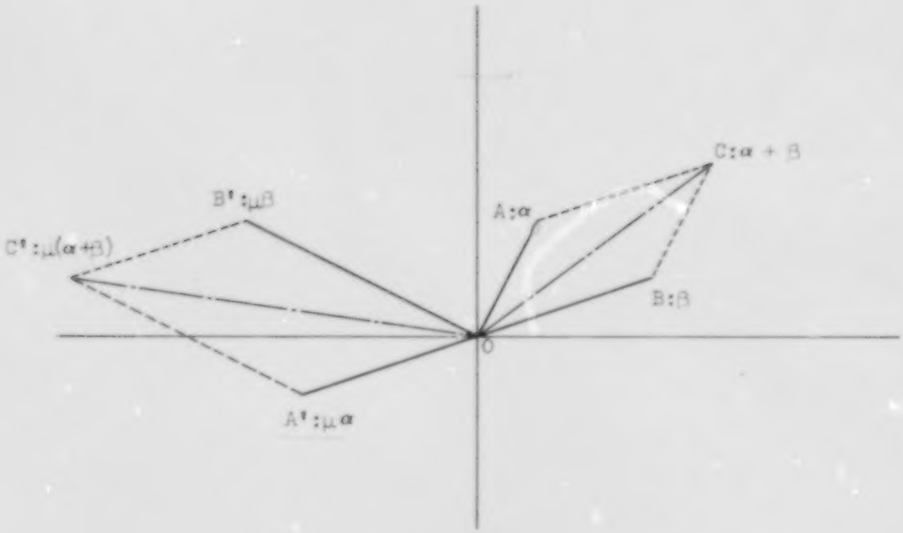

C':μ(α+β)
B':μβ
A:α
C:α + β
B:β
O
A':μα

Figure 2

which shows that the two triangles are similar. This implies that the angles OBC and OB'C' are congruent and so are their supplements, angles AOB and A'OB'. This completes the proof of the lemma.

Now for the proof of the theorem. Assume coordinates as follows:

$$A : \alpha = a + bi \neq 0 \;;\; B'' : \beta = c + di \neq 0 \;;\; B : \frac{1}{\beta} \;;\; B' : 1 \;;\; A' : \alpha\beta \,.$$

(See the figure). From the lemma we know that angles AOB and A'OB' are

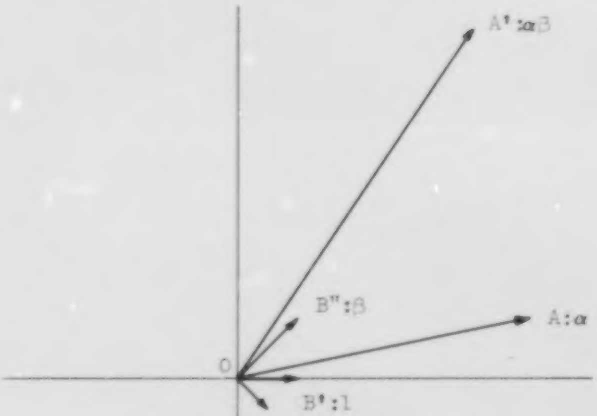

A':αβ
B'':β
A:α
O
B':1
B: $\frac{1}{\beta}$

Figure 3

congruent since we get the coordinates of A′ and B′ from A and B by multiplying by β. Furthermore, again from the lemma, angles BOB′ and B′OB″ are congruent since we get the coordinates of B′ and B″ from those of B and B′ by multiplying by β. Thus:

$$\angle BOB' = \operatorname{amp}\beta \ , \ \angle B'OA = \operatorname{amp}\alpha \ , \ \angle B'OA' = \operatorname{amp}\alpha\beta .$$

But

$$\angle BOB' + \angle B'OA = \angle BOA$$

and we have

$$\operatorname{amp}\beta + \operatorname{amp}\alpha = \angle BOA \cong \angle B'OA' .$$

This shows that the sum of the measures of the amplitudes of α and β is the measure of the amplitude of $\alpha\beta$. This completes the proof. Notice that in the figure we have made use of the restriction that a, b, c and d be positive. It is possible to extend the proof to establish the theorem without the restrictions imposed but we do not take the space for it here.

Theorem 2 can be used to prove two important formulas in trigonometry. For the benefit of those who know something of this subject, we now indicate how it goes. From Figure 1 it is clear that if r is the absolute value of the complex number $a + bi$, and θ its amplitude, then

$$\cos\theta = \frac{a}{r} \text{ and } \sin\theta = \frac{b}{r},$$

in other words,

$$a = r\cos\theta \text{ and } b = r\sin\theta .$$

Thus

$$a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

Similarly the complex number $c + di$ may be written:

$$c + di = s\,(\cos\phi + i\sin\phi)$$

where s is the absolute value and ϕ the amplitude.

Taking the product, we have

$$(a + bi)(c + di) = rs[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)] .$$

But rs is the absolute value of the product and it can be written

$$rs(\cos\gamma + i\sin\gamma)$$

where γ is the amplitude of the product. We know from Theorem 2 that $\gamma = \theta + \phi$. Hence we have the two formulas:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi .$$

These are two important formulas of trigonometry. In the usual treatment they are derived first and a result equivalent to Theorem 2 (De Moivre's Theorem) is shown to be a consequence.

Problems

1. We defined absolute value for real numbers. Show that this consistent with our definition of absolute value for complex numbers.
2. Find a condition that the vectors connecting the points $a + bi$ and $c + di$ to the origin be perpendicular.

References

6 (Appendix B), 12 (Chap. 7), 21 (Chap. 2).

Appendix 4

THE STRAIGHT LINE

1. Introduction

We assumed in various places in this book that all the points (x,y) which satisfy a given equation of the form ax + by + c = 0 lie on a straight line. Before proving this we should first recall a few geometrical facts. Two triangles ABC and A'B'C' are said to be similar if a 1 to 1 correspondence can be set up between the vertices of one and the other so that corresponding angles are congruent. That is, if the correspondence is: A ↔ A', B ↔ B', C ↔ C', then the following pairs of angles are congruent: A and A', B and B', C and C'. If the two triangles are similar then it is true that the lengths of corresponding sides are proportional. Thus if the correspondence is as indicated above and if the triangles are similar then:

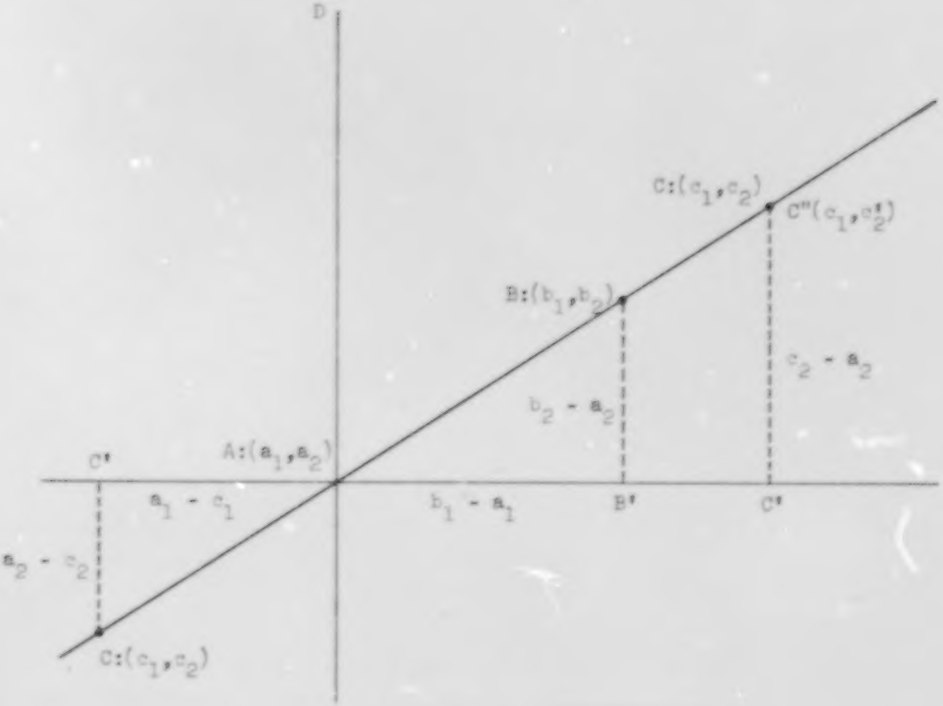
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

where AB indicates the length of the side $\overline{AB}$, etc. Conversely, if corresponding sides are proportional, the triangles are similar.

2. Slope.

We begin by defining a number, which we call s(A,B) associated with every pair of points of the plane not on a vertical line. We shall show that this number depends only on the line determined by the points and not on the points themselves. We will then justify our calling it the slope of the line. It is, in fact, the ratio of the rise to the horizontal distance for the line. Let A have the coordinates (a_1,a_2) and B the coordinates (b_1,b_2). We define

$$s(A,B) = \frac{b_2 - a_2}{b_1 - a_1}.$$



Notice what this means in terms of the figure. Since A and B are not on the same vertical line, $b_1 \neq a_1$ and the denominator of s(A,B) is not zero.

First, s(A,B) = s(B,A) since

$$\frac{b_2 - a_2}{b_1 - a_1} = \frac{a_2 - b_2}{a_1 - b_1}$$

as may be seen by multiplying the numerator and denominator of the left-hand fraction by -1. We now need the following theorem:

Theorem 1. If A, B and C are three collinear points, no two on the same vertical line, then

$$s(A,B) = s(A,C).$$

Proof: Because of the symmetry of s, we may assume without loss that $b_2 > a_2$. Let B' and C' be the feet of the perpendiculars from B and C respectively onto the horizontal line through A. Let the ray $\overrightarrow{AD}$ be perpendicular to the line $\overleftrightarrow{AB'}$ and above it, and let C have the coordinates (c_1,c_2).

If $\overleftrightarrow{AB}$ is horizontal, then $\overleftrightarrow{AC}$ is also. Furthermore $a_2 = b_2$, $a_2 = c_2$ and $s(A,B) = s(A,C) = 0$. Thus the theorem holds in this case.

Now the point B is either to the right of the ray $\overrightarrow{AD}$ or to the left of it. We consider the former case and leave the latter as a problem. The point C may be above or below the horizontal line through A. Hence we have two cases to consider.

First, suppose B is to the right of $\overleftrightarrow{AD}$ and C is above the horizontal line through A. Then, since we have taken B also to be above the line $\overleftrightarrow{AB'}$, we have

$$b_2 > a_2, \ c_2 > a_2 .$$

Furthermore, since $\overleftrightarrow{BC}$ intersects $\overleftrightarrow{AD}$ in the point A, then B and C are on the same side of $\overleftrightarrow{AD}$ and hence C as well as B is to the right of $\overleftrightarrow{AD}$. This means

$$b_1 > a_1, \ c_1 > a_1.$$

This implies the following equalities:

$$B'B = b_2 - a_2, \ AB' = b_1 - a_1, \ C'C = c_2 - a_2, \ AC' = c_1 - a_1.$$

Now the triangles AB'B and AC'C are similar for angle A is common, angles AB'B and AC'C are right angles and the third angles are congruent since the sum of the measures of the angles of a triangle is 180. Thus

$$\frac{C'C}{AC'} = \frac{B'B}{AB'} .$$

Using the expressions above for the distances, we see that this equality is equivalent to:

$$\frac{c_2 - a_2}{c_1 - a_1} = \frac{b_2 - a_2}{b_1 - a_1}.$$

Now the right side is s(A,B) and the left is s(A,C). Hence our proof is complete for this case.

Second, suppose B is to the right of $\overleftrightarrow{AD}$ and C is below the horizontal line through A. Then

$$b_2 > a_2 \text{ and } c_2 < a_2.$$

In this case B and C are on opposite sides of $\overleftrightarrow{AD}$, and hence C is to the left of $\overleftrightarrow{AD}$ and

$$b_1 > a_1, \ c_1 < a_1.$$

Then B'B and AB' have the same values in terms of the coordinates as before and

$$CC' = a_2 - c_2, \ AC' = a_1 - c_1.$$

However, the ratio $\frac{CC'}{AC'}$ is the same as before and we have again

$$s(A,B) = s(A,C).$$

Thus our proof is complete for the case in which B is to the right of the line $\overleftrightarrow{AD}$. As we noted above, we leave the other possibility as a problem.

To complete our discussion we now prove a converse of Theorem 1.

Theorem 2. If $s(A,B) = s(A,C)$, then the points A, B and C are collinear.

Proof: Using the same notation as for the previous theorem, let C" be the point of intersection of the line $\overleftrightarrow{CC'}$ with $\overleftrightarrow{AB}$. Then, by Theorem 1,

$$s(A,B) = s(A,C''),$$

and by the hypothesis of this theorem

$$s(A,B) = s(A,C).$$

Hence

$$s(A,C) = s(A,C'').$$

Since C" is on the vertical line through C, its first coordinate is the same as that of C, namely c_1. Call c_2' its second coordinate. Then the last equality can be written

$$\frac{c_2 - a_2}{c_1 - a_1} = \frac{c_2' - a_2}{c_1 - a_1} .$$

This implies $c_2 = c_2'$ and hence $C = C'$, which completes the proof.

Problem

Prove Theorem 1 for the case in which the point B is to the left of the line $\overleftrightarrow{AD}$.

3. The Equation, $rx + sy + t = 0$.

Here we assume that r, s, t are real numbers and not both r and s are zero. We need to show that all points (x,y) whose coordinates satisfy this equation lie on a straight line and that every point on this straight line has coordinates which satisfy the equation. In other words, we want to show that the set of points (x,y) whose coordinates satisfy the equation is the set of points on a straight line.

coordinates satisfy the equation is the set of points on a straight line.

First suppose p is a line which is not perpendicular to the x-axis. It has a slope which we may call m. Let (a,b) be some point on this line. Then for all points (x,y) on this line the slope is

$$\frac{(y - b)}{(x - a)} = m.$$

This equation is equivalent to

$$y - b = m(x - a)$$

or

$$y - mx + (ma - b) = 0,$$

if $x \neq a$ and is satisfied by the pair (a,b). This is the form of the given equation, where $s = 1$, $r = -m$ and $t = ma - b$. Hence we have shown that every point on the line p has coordinates which satisfy the equation.

Conversely suppose (x,y) is a point whose coordinates satisfy the equation $rx + sy + t = 0$ and consider the case when $s \neq 0$. Then this equation is equivalent to:

$$y - (-\frac{r}{s})x + \frac{t}{s} = 0.$$

If we let $m = -\frac{r}{s}$ and $\frac{t}{s} = -c$, the equation becomes

$$y = mx + c$$

which may be written $\frac{(y - c)}{(x - 0)} = m$. Thus for all points whose coordinates satisfy the equation, the slope of the line connecting (x,y) with $(0,c)$ is the same. Thus the set of points is a line.

Finally we need to consider the case when $s = 0$. Then the equation $rx + sy + t = 0$ reduces to $rx + t = 0$. Then if $r \neq 0$, this is equivalent to

$$x = -\frac{t}{r}.$$

This is a straight line parallel to the y-axis where every point of this line has first coordinate $-\frac{t}{r}$. Conversely, if a line is parallel to the y-axis it will have an equation $x = v$ which again is of the form desired. Hence our proof is complete.

Reference

8 (Chap. 7).

Appendix 5

MODULAR ARITHMETIC

1. *The Number System Modulo Twelve*.

Here we give briefly two examples of number systems which are quite different from any studied elsewhere in this book, though they have many properties in common with the familiar number systems. An important difference is that each contains only a limited number of elements but we shall discover some other differences as well. Notice that we here write of *number* systems - not numeral systems. The difference is not a matter of notation as in Chapter I.

On the faces of many clocks only the natural numbers from 1 to 12 inclusive appear. Each hour of the day is one of these numbers. Four hours after eleven is three. No matter how many hours we add to a given hour, our answer is one of the numbers of the set from 1 through 12 inclusive. So we shall consider a number system, S, containing just twelve numbers: 1, 2, 3, ..., 12 with a different kind of addition and multiplication. In this system

$$11 + 4 = 3, \; 7 + 8 = 3, \; 9 + 7 = 4.$$

The sum of two numbers of S is again a member of S. We get the sum in this system by adding in the set of integers, dividing by 12 and writing the remainder as the sum in this system. We call this the *number system modulo* 12. The addition table is easily constructed as follows:

Addition modulo 12

+	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	1
2	3	4	5	6	7	8	9	10	11	12	1	2
3	4	5	6	7	8	9	10	11	12	1	2	3
4	5	6	7	8	9	10	11	12	1	2	3	4
5	6	7	8	9	10	11	12	1	2	3	4	5
6	7	8	9	10	11	12	1	2	3	4	5	6
7	8	9	10	11	12	1	2	3	4	5	6	7
8	9	10	11	12	1	2	3	4	5	6	7	8
9	10	11	12	1	2	3	4	5	6	7	8	9
10	11	12	1	2	3	4	5	6	7	8	9	10
11	12	1	2	3	4	5	6	7	8	9	10	11
12	1	2	3	4	5	6	7	8	9	10	11	12

This system satisfies the usual properties for addition:

1. Closure. (The sum of two numbers in S is in S.)
2. Commutativity. (Since $a + b = b + a$, the remainders when these two numbers are divided by 12 are the same.)
3. Associativity. (We assume this, though it is not hard to prove).
4. Existence of an identity element. Here the additive identity is 12 because $12 + a = a = a + 12$. (We could make the analogy closer by replacing 12 by 0.)
5. Existence of an additive inverse. If a is a number S, than $12 - a$ is also a number of S and it is the additive inverse because

$$(12 - a) + a = 12,$$

which is the additive identity.

We can summarize this by saying that the set of numbers modulo 12 forms an abelian group under addition. (See Section 7.6.)

We can also define multiplication in the system modulo 12. To find a product in this system, we first find the product in the set of integers and call the product modulo 12, the remainder when the ordinary product is divided by 12. For instance 7 • 8 = 8 in the system modulo 12 because 7 • 8 = 56 and the remainder when 56 is divided by 12 is 8. By means of this definition, the product of any pair of numbers of S is again a member of S. Here the multiplication table is:

Multiplication modulo 12

X	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	10	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

It is easy to verify that multiplication also has the first four properties which we listed for Addition. Here the multiplicative identity is 1. But when it comes to a multiplicative inverse, we run into trouble. We can see from the table that no multiple of 8 is 1 (there is no 1 in the 8 row) and hence 8 has no multiplicative inverse. To consider the general case, suppose a number b in S has a multiplicative inverse. Then for some integers x and y we must have

$$bx = 12y + 1 .$$

This is equivalent to

$$bx - 12y = 1.$$

Let g denote the g.c.f. of b and 12. By Section 4.2, x and y integers implies that g is a factor of 1. Hence unless $g = 1$ the equation has no solution. On the other hand, if $g = 1$, Theorem 4 of Section 4.4 shows that there is a solution. Thus, for this system, the only numbers which have multiplicative inverses are those having no factors greater than 1 in common with 12, that is:

$$1, 5, 7, 11.$$

This can also be verified from the table.

Notice that from the table, $12a = 12$ for all numbers a of S. Thus 12 has another property of zero. The reader may discover many interesting properties of this table.

2. The Number System Modulo Seven

Another example of a finite number system is given by the seven days of the week. If we number them:

$$0, 1, 2, 3, 4, 5, 6,$$

where 0 corresponds to Sunday, 1 to Monday, etc., then, since Tuesday is three days after Saturday, we have

$$6 + 3 = 2.$$

Here we find the sum by taking the remainder after dividing by seven. As above we can construct the addition and multiplication tables as follows:

Addition Modulo 7

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Multiplication Modulo 7

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Just as before we can see that the set of numbers T, modulo 7, has the properties of addition which make it an Abelian group. And also the first four properties hold for multiplication as for the system modulo 12. But then there is a fundamental difference. Suppose, as before, that b is an element of T; then it will have a multiplicative inverse if and only if

$$bx = 7y + 1$$

is solvable. This equation is equivalent to

$$bx - 7y = 1.$$

Here 1 is the g.c.f. of b and 7 unless b = 0, because 7 is a prime number. All the natural numbers less than 7 satisfy the required condition. So for this number system we have a multiplicative inverse except for zero. For both these systems, the distributive property holds. Thus the number system modulo 7 is a field (see Section 7.9).

3. Conclusion.

It is easy to see from the above that the number system modulo m forms a field if and only if m is a prime number. If m is composite, we not only fail to have a multiplicative inverse in many cases, but we also have products of two non-zero numbers equal to zero. For instance, in the system modulo 12, $6 \cdot 4 = 0$. Thus the number system modulo 12, does not even form an integral domain (see Section 7.6). Thus, it appears that a number system modulo m is not an integral domain unless it is a field. It turns out to be true that every integral domain with a finite number of elements is a field.

It can also be shown that if F is a field with a finite number of elements then the number of elements in F is a power of a prime number. However, it should be noted that the numbers modulo m where $m = p^2$ for instance, p a prime number, is not a field since in this system the number p has no multiplicative inverse. One has to use a different method of construction when m is a power of a prime number but not a prime number. This method may be found in some of the references below.

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Numbers 10 and 15.

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. . . .

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In many of the books in the list above may be found additional references.

Answers to Problems

Section 2.7

1. The first card will contain those numbers whose last digit in the binary system is 1, that is, the odd numbers: 1, 3, 5, 7, ..., 29, 31. The second card will contain those whose next to the last digit is 1. Thus the binary representation will be of the form: abc1d where the letters a, b, c, d are arbitrary. This number is: $a(\text{two})^4 + b(\text{two})^3 + c(\text{two})^2 + 1(\text{two}) + d$, where each of a, b, c, d is either 1 or 0. Thus the number will be some multiple of 4 plus 2 plus either zero or 1. In other words the numbers on the second card will be of the form: $4n + 2 + d$ where $d = 0$ or 1. That is, the form must be $4n + 2$ or $4n + 3$. So on the second card the numbers will be: 2, 3; 6, 7; 10, 11; 14, 15; 18, 19; 22, 23; 26, 27; 30, 31 where the first two are for $n = 0$, the second two for $n = 1$, etc.

   The third card will contain those whose third from the last digit is 1, that is those numbers of the form:

   $$8n + 4 + e$$

   where e is a number between 0 and 3 inclusive: thus the numbers

   $$8n + 4,\ 8n + 5,\ 8n + 6,\ 8n + 7.$$

   Here the numbers will be:

   4, 5, 6, 7; 12, 13, 14, 15; 20, 21, 22, 23; 28, 29, 30, 31

   where again the grouping is according to the value of n.

   The fourth card will contain the numbers which, in the binary system have second digit 1, that is those of the form $16n + 8 + f$ where f is a number between 0 and 7 inclusive, that is, those of the forms:

$16n+8$, $16n+9$, $16n+10$, $16n+11$, $16n+12$, $16n+13$, $16n+14$, $16n+15$.
Here the numbers will be then:

8, 9, 10, 11, 12, 13, 14, 15 ; 24, 25, 26, 27, 28, 29, 30, 31.

Finally, the fifth card will contain the numbers from 16 to 31 inclusive.

2. If one is to weigh one ounce, he must have a weight of 1 ounce, and for two he must have a weight of two ounces. Now, by combining the two he can weigh three ounces. He needs a four-ounce weight and with this he can by various combinations weigh 4, 5, 6, 7 ounces. He needs an eight ounce weight and with this he can weigh up to 15 ounces. So this may be continued. Weights are needed for the powers of 2; for a given weight those which are used correspond to the digit 1 in the binary system, much the same as in the problem above.

3. The weights of the cups will be $\frac{1}{2}$, 1, 2, 4, 8 ounces respectively. These, together with the innermost weight will suffice to weigh all multiples of one-half up to and including 16 ounces, that is, one pound.

Section 3.2

1. Suppose $r > s$ and $t > s$, the fourth case. Then if $r > t$, t is between r and s. If $t > r$, then r is between t and s.

2. Since a is between b and c, then either

   i) $b < a < c$ or ii) $c < a < b$.

   Since c is between a and d, then either

   iii) $a < c < d$ or iv) $d < c < a$.

   Thus if $c < a$, conditions ii) and iv) must hold and $d < c < a < b$. If $c > a$, conditions i) and iii) hold and $b < a < c < d$. In both cases a and c are between d and b by the transitive property.

3. By the definitions we have:

$$A\ R\ B \text{ and } A\ R\ C;\ B\ R\ C \text{ and } B\ R\ D;\ C\ R\ A \text{ and } C\ R\ D.$$

This contradicts Property 1 of an order relationship since A R C and C R A. Between A and C there are two elements, B and D; between B and A there are C and D. Similarly we have such relationships for all pairs. Hence this set has the property of betweenness as defined above. It is also dense because between any two there is a third.

Section 3.4

1. If an element is in A but not in B, it is counted once on both sides of the equation given. If an element is in both A and B it is counted twice on both sides. Hence the equality holds.

2. One interpretation would be: if $n(A) = n(C)$ then

$$n(A) + n(B) = n(C) + n(B).$$

3. Consider the arrays

(... ..)   (... ..)   (... ..)

where the number of dots in the respective arrays is a, b and c. The associative property of addition is illustrated by the fact that we may count first the number of dots in the first two arrays and then those in the last, or we may count the number of dots in the last two and add this to the number of dots in the first. In both cases we have the same number of dots.

Section 3.5

1. 1) $ab + c = ab + ac$ would imply $c = ac$. (We are here using the cancellation property in Section 3.7 but this property is familiar to

you.) Now $c = ac$ if $c = 0$. Otherwise a must be equal to 1. Hence the only two possibilities are: $c = 0$ or $a = 1$ or both.

2) $(ab)(ac) = a(b(ac)) = a((ac)b) = (a(ac))b = ((a^2)c)b = a^2(cb)$ using the commutative and associative properties for multiplication. Then $a^2(bc)$ will be equal to $a(bc)$ if $a = 0$ or $a = 1$ or $bc = 0$.

3) $(ab) + (ca) = (ab) + (ac) = a(b + c)$ by the commutative and distributive properties for all values of a, b and c.

2. For the proof we may start with the left side of the equation: $a(b + c + d) = a(\underline{b + c} + d)$ where the underline indicates that $b + c$ is to be thought of as single number, using the associative property for addition. Then $a(\underline{b + c} + d) = a(b + c) + ad$ by the distributive property. Using this property again we have $(ab + ac) + ad$. By the associative property this is equal to $ab + ac + ad$.

3. Now $23 \cdot 78 = (20 + 3)(70 + 8) = (20 + 3)70 + (20 + 3)8 = 20 \cdot 70 + 3 \cdot 70 + 20 \cdot 8 + 3 \cdot 8$, using the distributive property twice and the associative property for addition. This then is equal to:

$$1400 + 210 + 160 + 24.$$

By the decimal notation and the commutative property this is equal to:

$$1400 + 200 + 100 + 10 + 60 + 20 + 4.$$

Using the distributive property we have

$$(14 + 2 + 1)(100) + (1 + 6 + 2)10 + 4$$

which is 1794. For the product $78 \cdot 23$ the order of numbers in each product is reversed.

4. Dividing by 7, then by 11 and finally by 13 is equivalent to dividing by $7 \cdot 11 \cdot 13 = 1001$. On the other hand:

$$327{,}327 = 327{,}000 + 327 = 327(1000) + 327 \cdot 1 = 327(1000 + 1) = 327 \cdot 1001.$$

The same would hold for the number abc, abc.

Section 3.6

1. If $b > c$, then $b = c + x$ for some counting number x. Then $b - c = x$. But, by the well-defined property for multiplication, $b = c + x$ implies $ba = ca + xa$ which gives us on the one hand that

$$ba - ca = xa$$

and on the other that $(b - c)a = xa$.

2. Let $b - c = x$; we know that x is a counting number. Then

$$a + (b - c) = a + x$$

by the well-defined property for addition. Now, $x + c = b$. Hence

$$a + (b - c) + c = a + x + c = a + b.$$

Hence, if we add c to $a + (b - c)$ we get $a + b$. This means that $a + (b - c)$ must be equal to $(a + b) - c$.

3. Here the equality is not true, for let $a = 5$, $b = 3$, $c = 1$. Then $a - (b + c) = 5 - (3 + 1) = 5 - 4 = 1$ while

$$(a - b) + c = (5 - 3) + 1 = 2 + 1 = 3.$$

Section 3.7

1. Here $ca = cb$ implies $ca - cb = 0$, that is

$$c(a - b) = 0.$$

But the product of two whole numbers can be zero only if one is zero. Hence $c = 0$ or $a - b = 0$. Thus, if $c \neq 0$, $a = b$.

Section 3.8

1. $\frac{c}{\left(\frac{b}{a}\right)} = c\left(\frac{a}{b}\right) = \frac{ca}{b}$, while, $\frac{\left(\frac{c}{b}\right)}{a} = \frac{c}{ba}$. These two fractions will be equivalent only if:

$$(ca)(ba) = bc,$$

that is

$$a^2(bc) = (bc).$$

Thus the equality will hold only if $bc = 0$ or $a^2 = 1$. For instance, one set of values for which the equality does not hold is $c = 12$, $b = 6$, $a = 2$. Here the left side is equal to 4 and the right to 1.

2. Here since $\frac{c}{b} + \frac{c}{a} = \frac{(ca + bc)}{ab}$, the two fractions are equivalent only if

$$abc = (b + a)(ca + bc) = abc + a^2c + b^2c + abc.$$

Since all the letters stand for whole numbers, the only possibility for equality is that a^2c, b^2c and abc are all zero. This will happen if c is zero or if both a and b are zero. One example of inequality is for $a = b = c = 1$ when the left side is equal to $\frac{1}{2}$ and the right to 2. If $a = 0$ or $b = 0$ the given fractions have no meaning.

3. This equality holds for all numbers by the distributive property:

$$\frac{c}{a} + \frac{b}{a} = c\left(\frac{1}{a}\right) + b\left(\frac{1}{a}\right) = \frac{(c + b)}{a}.$$

4. Here let $c - b = x$, a whole number. This means that $c = b + x$. Thus $\frac{c}{a} = \frac{(b + x)}{a} = \frac{b}{a} + \frac{x}{a}$ and $\frac{(c - b)}{a} = \frac{x}{a}$. The previous equality then shows that if we add $\frac{b}{a}$ to $\frac{x}{a}$ we get $\frac{c}{a}$ which means that $\frac{x}{a}$ is equal to $\frac{c}{a} - \frac{b}{a}$.

Section 3.9

1. The inequality $a + c < b + c$ implies the existence of a counting number x such that $(a + c) + x = b + c$. Then, from the associative and commutative properties: $(a + x) + c = b + c$. The cancellation property for addition shows us that $a + x = b$ which is equivalent to $a < b$.

2. Here an indirect proof seems better. We know that if $a < b$ is false then one of two things can happen: $a = b$ or $a > b$. In the former case $ac = bc$ and in the latter $ac > bc$. Each of these denies $ac < bc$. Hence $a < b$ cannot be false, thus is true. The same kind of argument could have been used in the solution of Problem 1.

Section 4.2

1. Suppose in $b = cq + r$, b and c are counting numbers, q is a whole number and r is a whole number less than c. Then the point corresponding to b on the number line will be between that corresponding to cq and $c(q + 1)$. In fact it will be r units to the right of cq. Graphically to determine q and r, one can "lay off" the number of units corresponding to c again and again until we have a point to the right of that corresponding to b. The multiple before that will give the number q and r will be $b - cq$. If b is a multiple of c, of course r will be zero.

2. Property 1: r a factor of s implies $rx = s$ for a counting number x. Also s a factor of r implies $sy = r$ for some counting number y. Then, by the well-defined property for multiplication we may replace r by sy in the equation $rx = s$ and have

$$(sy)x = s.$$

By the associative property we then have

$$s(yx) = s.$$

Since s is not zero, the cancellation property for multiplication implies $yx = 1$. It is rather obvious that if the product of two whole numbers is 1, they both must be 1, but we can prove it using the properties which have been discussed. Suppose $y \neq 1$. Then $y \geq 2$ which implies $xy \geq 2x$. But $x \geq 1$ implies that $2x \geq 2$ by the well-defined property for multiplication of inequalities. This means that $yx \geq 2$, which is false. Thus $y = 1$ and $x = 1$.

Property 2. Here r a factor of s and s a factor of t implies that $rx = s$ and $sy = t$ for counting numbers x and y. Thus, $(rx)y = t$, or $r(xy) = t$ which shows that r is a factor of t.

Property 3. Here $rx = s$ and $ry = t$ implies

$$s + t = rx + ry = r(x + y)$$

$$st = rx \cdot ry = r(rxy).$$

For the final case, suppose $s - t = u$. Then $rx - ry = u$, and by the Section 3.6, $r(x - y) = u$ which shows that r is a factor of u.

3. Using the results of this section, we see that for counting numbers x and y we have: $x = sq + r$, $y = sq' + r'$ where r and r' are the remainders when x and y are divided by s. Then

$$x - y = sq + r - sq' - r' = s(q - q') + (r - r').$$

If $r = r'$, then $x - y = s(q - q')$ which shows that $x - y$ is divisible by s.

4. We can use the same notation here as in the previous problem. But in this case $x - y$ is given to be a multiple of s. Then by Property 3 above $r - r'$ must be a multiple of s. If $r > r'$ then $r - r'$ is a whole number less than s, since r is less than s. But the only multiple of s which is less than s is zero; hence in this case $r - r' = 0$, or $r = r'$. If $r < r'$ the same argument can be given. This completes the proof.

Section 4.3

Here it is perhaps more instructive to consider the general case in terms of letters. Suppose n is a counting number which is equal to the product of two counting numbers t and u. Let r be the square root of n, that is, the number (we assume the existence of such a number) so that $r^2 = n$. We can show that if $t > r$, then $u < r$, in other words, if n has a prime factor greater than r, it has one less than r. To prove this suppose

$$t > r \text{ and } u > r.$$

Then $tu > r \cdot r = n$ which contradicts $tu = n$. Thus if $t > r$, $u < r$ and if t is a prime number, u is either a prime less than r or has a prime factor less than r. This means that if we are searching for prime factors of a given number we need only look as far as the square root of the number. In the case of 4501, $67^2 < 4501$ and $68^2 > 4501$. Hence if 4501 has no prime factor less than 68 it is a prime number.

Section 4.4

1. Given two sets, A and B. There is one common subset which any pair of sets has, the empty set, or null set. Hence two sets having no elements in common is the situation which corresponds to two numbers having only the common factor 1.

2. The same argument holds here as for the usual euclidean algorithm. Any common factor of 299 and 221 will be a factor of 78, any common factor of 221 and 78 will be a factor of 13. Conversely, 13 is a factor of 78 and, by the second equation, is a factor of 221; by the first equation it is also a factor of 299.

3. Here the calculation is:

$$89 = 1 \cdot 55 + 34$$

$$55 = 1 \cdot 34 + 21$$

$$34 = 1 \cdot 21 + 13$$

$$21 = 1 \cdot 13 + 8$$

$$13 = 1 \cdot 8 + 5$$

$$8 = 1 \cdot 5 + 3$$

$$5 = 1 \cdot 3 + 2$$

$$3 = 1 \cdot 2 + 1$$

$$2 = 1 \cdot 1 + 1$$

Here all the quotients are 1. This means that the process is as long as it can be for the size of the numbers involved.

4. Suppose g divides a and b. Then by the divisibility properties, g divides r. Conversely if g divides r and b, it divides a. Hence the common factors of a and b are the same as the common factors of b and r. This shows that the g.c.f. is the same for both. Hence, in the process of the euclidean algorithm, the g.c.f. of the dividend and divisor is the same in all the equations. Hence in the last step, the divisor is the g.c.f. and hence is the g.c.f. of all pairs, including the original pair of integers.

5. If g is the g.c.f. of a and b, then we might write $a = ga'$ and $b = gb'$, where, by Theorem 2 of this section, 1 is the g.c.f. of a' and b'. Then the following three equations are equivalent:

$$ax + by = g, \qquad ga'x + gb'y = g, \qquad a'x + b'y = 1$$

and the last equation is solvable in integers by Theorem 4.

Section 4.5

1. The l.c.m. for two numbers can be considered to correspond to the union of two sets. The union is the smallest set which includes both.

2. Using the zero exponents as in this section we see that

$$525 \cdot 4455 = (3 \cdot 5^2 \cdot 7 \cdot 11^0)(3^4 \cdot 5 \cdot 7^0 \cdot 11)$$

and

$$(\text{g.c.f.}) \cdot (\text{l.c.m.}) = 3 \cdot 5 \cdot 7^0 \cdot 11^0 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11.$$

For the second equation we have just a rearrangement of the exponents. For the first part, that is, the g.c.f., we use the smaller of the two exponents for each prime and for the second part, that is, the l.c.m., we use the larger. From this it can be seen that the two products will be always the same.

To shorten the process we could find the g.c.f. of 525 and 4455 by the euclidean algorithm. Then divide this g.c.f., 15, into each of the numbers, having:

$$525 = 15 \cdot 35 \text{ and } 4455 = 15 \cdot 297.$$

Then, using the above result, we have

$$\begin{aligned} 525 \cdot 4455 &= (15 \cdot 35)(15 \cdot 297) = (\text{g.c.f.})(\text{l.c.m.}) \\ &= 15(\text{l.c.m.}). \end{aligned}$$

This imples that the l.c.m. is equal to

$$15 \cdot 35 \cdot 297.$$

From this point we can proceed as above. Doing it this way we do not need to factor any of the numbers involved.

3. It may be verified that 3 is a common factor of 23,082 and 155,925. Since the second is divisible by 9 but not the first (we could use results of the next section here) 3 is the highest power of this number which divides both. Looking at the factorization of 155,925 we see that the only other possible prime common factors are 5, 7 and 11, with 5 easily excluded. Now

$$23{,}082 = 71 \cdot 297 + 57 \cdot 35.$$

The number 7 divides the second term on the right but not the first, while 11 divides the first but not the second. Hence neither 7 nor 11 can divide 23,082. This shows that 3 is the g.c.f. of the numbers 23,082 and 155,925.

4. If bell A rings every 12 minutes, it will ring at times 12t minutes after noon for any whole number value of t. Similarly bell B will ring every 15u minutes after noon. If they ring together we would have

$$12t = 15u,$$

that is,

$$4t = 5u.$$

This means that t must be divisible by 5 and the least such counting number t is 5 itself. Hence they will both ring together $12 \cdot 5 = 60$ minutes after noon, that is, first at one-o'clock and each hour thereafter.

5. For this, using the same notation, bell A will ring at 12t minutes after noon and bell B, 15u + 1. Then the equation is

$$12t = 15u + 1.$$

This is impossible because $12t - 15u$ is divisible by 3 but 1 is not. Hence the bells will never ring together. There is a further question in this case. Suppose both bells ring together at noon and each at regular intervals thereafter. In all cases will they ring together again sometime?

6. Replacing 15 by 35 would give the equation

$$12t = 35u + 1.$$

This equation is equivalent to

$$12t - 35u = 1.$$

By Theorem 4 of Section 4.4 this has solutions. It can be seen by trial that one solution is $t = 3$, $u = 1$.

To find the other possibilities consider the two equations:

$$12t - 35u = 1$$

$$12 \cdot 3 - 35 = 1.$$

Subtracting the second from the first we have

$$12(t - 3) - 35(u - 1) = 0$$

which is equivalent to

$$12(t - 3) = 35(u - 1).$$

Since 1 is the g.c.f. of 12 and 35, $t - 3$ must be divisible by 35, that is

$$t - 3 = 35n.$$

Replacing $t - 3$ by $35n$ we have

$$12 \cdot 35n = 35(u - 1)$$

or

$$12n = u - 1.$$

Combining these results we have

$$t = 3 + 35n, \quad u = 1 + 12n.$$

When $n = 0$ we have the values we started with: $t = 3$ and $u = 1$; when $n = 1$ we have the next pair: $t = 38$, $u = 13$. So we may get all the times when the two bells ring together.

7. Suppose the times which bell A ring are $an + c$ minutes after noon where $n = 0, 1, 2, 3, \ldots$ and for B the times are $bm + d$ where $m = 0, 1, 2, \ldots$. Then we consider the equality

$$an + c = bm + d.$$

If a and b have a g.c.f. greater than 1, it must divide $d - c$. It is true that in this case there is always an infinity of solutions. Proofs of this may be found in books on the theory of numbers.

Section 4.6

1. The five digit number abcde denotes the following:

$$a(10^4) + b(10^3) + c(10^2) + d(10) + e.$$

This can be written

$$a(9999 + 1) + b(999 + 1) + c(99 + 1) + d(9 + 1) + e =$$

$$9999a + 999b + 99c + 9d + (a + b + c + d + e).$$

Thus the given number differs from $(a + b + c + d + e)$ by a multiple of 9, which means that the remainders when the number and the sum of its digits are divided by 9 are the same. (See Problems 3 and 4 of Section 4.2.)

2. Consider the same five digit number as above except that in the second line we write it in a little different way:

$$a(10^4) + b(10^3) + c(10^2) + d(10) + e =$$

$$a(9999 + 1) + b(1001 - 1) + c(99 + 1) + d(11 - 1) + e.$$

Our reason for writing it this way is that the numbers 9999, 1001, 99, 11 are all multiples of 11. Thus, instead of the sum of the digits we use the expression

$$a - b + c - d + e.$$

It is this which has the same remainder when divided by 11 as the given number.

There is an alternative test, writing the number in another way, as follows.

$$a(9999 + 1) + (10b + c)(99 + 1) + (10d + e).$$

This differs from $a + (10b + c) + (10d + e)$ by a multiple of 11. Here we add the digits two at a time. For instance, if the number is 23457. Then we form the sum: $57 + 34 + 2 = 93$. This means that the remainders are the same when 93 and 23457 are divided by 11.

3. The remainder when a number is divided by 9 depends only on the digits-- not on the order in which they are written. Thus the two numbers of the trick have the same remainders when divided by 9. This means that their difference is a multiple of 9, that is, the sum of the digits of the difference is a multiple of 9. This knowledge will allow one to determine the missing digit of the trick unless the sum of all but one turns out to be a multiple of 9. In this case one can only guess whether the missing digit is 0 or 9.

4. The number 3 has the same test for divisibility since 10 is 1 more than a multiple of 3 as well as 1 more than a multiple of 9.

5. Since 7 is 6 + 1, it will have the same characteristics for base 6 and divisibility as 11 has for base 10.

6. If r is the last digit of a numeral to the base 6, the number will be of the form

$$6n + r$$

where n is a counting number. Thus if one is to determine by looking at r whether or not there is divisibility, this will be possible only for divisors of 6. Thus if $r = 0$, 2 or 4 we know $6n + r$ is even; if $r = 0$ or 3 we know that $6n + r$ is divisible by 3; if $r = 0$ we know that $6n + r$ is divisible by 6.

Section 4.8

1. If $n = 1$, then $N = 2 + 1 = 3$ and N is the next prime after 2. If $n \neq 1$, we have, $n \geq 2$ and thus

$$p_1p_2 \cdots p_{n-1} \geq 2$$

since $p_1 = 2$. Thus

$$N = (p_1p_2 \cdots p_{n-1})p_n + 1 > 2p_n.$$

Hence, by Bertrand's postulate, there is a prime number between p_n and N. This shows that N cannot be the (n + 1)th prime number.

2. Each of the ten numbers exhibited is divisible by some one of the first five primes. For instance, n + 7 is divisible by 7 since n is. Consider a typical member of the set:

$$n + i, \quad 2 \le i \le 11.$$

Each i is divisible by some prime between 2 and 11, n is divisible by all the primes between 2 and 11; hence n + i is divisible by any prime which divides i. Also n + 12 is composite since n is divisible by 3. This example is more efficient than that given in the section because

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 < 11!$$

Section 4.9

1. The set S does not have the property described in the problem. A fundamental reason is that if it did hold, there would be a euclidean algorithm and this in turn implies a g.c.f. and the fundamental theorem of arithmetic. But a simpler way to show that it does not hold is to give an example. Let a = 60 and b = 8. Since 7 is not in the set, the largest q of the set for which $8q \le 60$ is q = 6. For this q,

$$60 = 6 \cdot 8 + 12$$

and 12 is greater than 8.

2. First, the units are those numbers $\frac{a}{b}$ in simplest form for which $\frac{b}{a}$ is also in the set. This means that the units are the fractions of the form $\frac{a}{b}$ where both a and b are odd integers. Thus the numbers of the set will be:

$$u_1,\ 2u_2,\ 2^2u_2,\ 2^3u_3,\ \ldots,\ 2^s u_s,\ \ldots$$

where u_i are units. From this it may be seen that the primes are

$$2u$$

where u is a unit. Furthermore, if $2^k u$ is a number of the set it can be written as a product of prime numbers as follows:

$$(2u_1)(2u_2)(2u_3)\ \dots\ (2u_k)$$

and this decomposition is unique except for changing the units.

Section 5.3

1. Suppose $\frac{a}{b} = \frac{c}{b}$, that is, the two fractions are equivalent. Then by the definition we must have $ab = bc$. But b is different from zero and hence by the cancellation property of multiplication, $a = c$. Also if $a = c$ then the two fractions are equivalent because then, by the well-defined property for multiplication, $ab = cb$.

2. To show that the answer is "yes" when the words "numerator" and "denominator" are interchanged in the previous problem we would start with $\frac{a}{b} = \frac{a}{c}$ and use exactly the same procedure.

Section 5.4

1. First we want to show that

$$\frac{r}{s} \text{ and } \frac{r}{s}$$

are equivalent fractions. By Theorem 1 of Section 5.3 this is true since $rs = sr$.

Second we need to verify that

$$\frac{a}{b} \text{ equivalent to } \frac{c}{d} \text{ implies } \frac{c}{d} \text{ equivalent to } \frac{a}{b}.$$

Again, using Theorem 1 of Section 5.3, we see that the equivalence of the first pair of fractions implies $ad = bc$ which, in turn, implies the equivalence of the second pair.

Finally we need to show that:

$\frac{a}{b}$ equivalent to $\frac{c}{d}$ and $\frac{c}{d}$ equivalent to $\frac{r}{s}$, implies $\frac{a}{b}$ and $\frac{r}{s}$ are equivalent.

The given equivalences imply:

$$ad = bc \text{ and } cs = rd.$$

Then, using the well-defined property of multiplication we have

$$ads = bcs = brd$$

and the cancellation property implies $as = br$ which, in turn, implies

$$\frac{a}{b} \text{ is equivalent to } \frac{r}{s}.$$

2. If there is a classification satisfying Properties 1e, 2e, and 3e, then we call two elements equivalent if they are in the same class. 1e shows that A is equivalent to itself, 2e that if A and B are equivalent so are B and A, and 3e shows the transitive property of equivalence.

3. If the set S is the set of whole numbers and R is $\leq$, then the relationship is reflexive and transitive but not symmetric.

Let S be the set of all people and R the relationship of having one parent in common. Then R is reflexive and symmetric but not necessarily transitive, for a could have father A and mother B, b could have father A and mother C, while c could have mother C and father D. Thus a and c would not have a common parent.

Finally, let S be the set of whole numbers and let $a \; R \; b$ mean that $a = 0a$ and $b = 0b$, that is, both a and b are zero. Then $1 \; R \; 1$ is false. But $a \; R \; b$ implies that a and b are zero and hence b and a are zero and $b \; R \; a$. For the transitive property we have two hypotheses: a and b are zero, b and c are zero. Hence a and c would be zero. So this set with the R defined has the symmetric and transitive properties but not the reflexive property.

Section 5.5

1. If b = 0 the left side of the equation $(\frac{b}{a})(\frac{d}{c}) = (\frac{bd}{ac})$ becomes $(\frac{0}{a})(\frac{d}{c}) = 0(\frac{d}{c}) = 0$ and the right side becomes $(\frac{0}{ac}) = 0$. Hence the equality holds here also.

Section 5.6

1. $$\frac{a}{b}\cdot\frac{bc}{ad} = \frac{abc}{bad} = \frac{c}{d}$$

because abcd = cbad.

Section 5.7

1. If b = 0, then $(\frac{b}{a}) + (\frac{d}{c}) = 0 + (\frac{d}{c}) = \frac{d}{c}$. Also $\frac{(bc + ad)}{ac}$ becomes $\frac{ad}{ac}$ which is equal to $\frac{d}{c}$. The proof is similar if d = 0 or b = d = 0.

Section 5.8

1. Here we wish to prove that $(\frac{a}{b})[(\frac{c}{d})(\frac{e}{f})] = [(\frac{a}{b})(\frac{c}{d})](\frac{e}{f})$. The left side is equal to $(\frac{a}{b})(\frac{ce}{df}) = \frac{ace}{bdf}$ by the definition of multiplication of fractions and the associative property of multiplication. The right side will be equivalent to the same fraction.

2. Here the need is to prove

$$\frac{r}{s} + (\frac{t}{u} + \frac{x}{y}) = (\frac{r}{s} + \frac{t}{u}) + \frac{x}{y}.$$

It was proved above that the left side is equal to $\frac{(ruy + sty + sux)}{suy}$.

The right side becomes

$$\frac{(ru + st)}{su} + \frac{x}{y} = \frac{(ruy + sty + xsu)}{suy},$$

which by the associative properties of addition and multiplication is equivalent to the previous fraction.

3. Here we wish to prove that if $\frac{a}{b} = \frac{c}{d}$, then $(\frac{a}{b})(\frac{r}{s}) = (\frac{c}{d})(\frac{r}{s})$. The last equality may be written

$$\frac{ar}{bs} = \frac{cr}{ds}\ .$$

We then need to show

$$ards = crbs.$$

But $ad = cb$, since $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent fractions. Hence, by the well-defined property of multiplication, $ards = crbs$ which shows the equivalence of the two given fractions above.

4. Here we wish to prove:

$$\frac{a}{b} + \frac{c}{d} = \frac{e}{f} + \frac{c}{d}$$

implies that

$$\frac{a}{b} = \frac{e}{f}\ .$$

We then have $\frac{(ad + bc)}{bd} = \frac{(ed + cf)}{fd}$ or

$$(ad + bc)fd = bd(ed + cf)$$

$$afd^2 + bcfd = bed^2 + bdcf.$$

By the associative and commutative properties of multiplication and the cancellation property of addition for whole numbers this implies

$$afd^2 = bed^2.$$

Since $d^2 \neq 0$, $af = be$ and the fractions $\frac{a}{b}$ and $\frac{e}{f}$ are equivalent.

Section 5.10

1. As in the section we take $r = \frac{b}{a}$ and $s = \frac{d}{c}$. If we write them as fractions with the same numerators we have

$$r = \frac{bd}{ad} \text{ and } s = \frac{bd}{cb}$$

and our equation becomes:

$$\frac{bd}{ad} + x = \frac{bd}{cb}.$$

Here we need to use a slightly different technique to eliminate the fractions. The given equality will be equivalent to that obtained by multiplying both sides by the product abcd. This gives:

$$(\frac{bd}{ad})(abcd) + x(abcd) = (\frac{bd}{cb})(abcd)$$

or

$$(bd)(bc) + x(abcd) = (bd)(ad).$$

If we divide both sides by (bd) we have an equivalent equation

$$bc + x(ac) = ad.$$

This is the same equation as (1) above and the rest of the proof is the same.

However, notice that

$$r = \frac{b}{a} = \frac{bd}{ad} > s = \frac{d}{c} = \frac{bd}{bc}$$

if and only if

$$ad < bc.$$

Hence if two fractions with the same numerator are to be compared, the lesser fraction has the greater denominator; while if the denominators are equal, the lesser fraction has the lesser numerator.

2. First we show that $\frac{(b + c)}{(a + d)} < \frac{c}{d}$. Using the results of Section 5.9 we see that this inequality is equivalent to

$$(b + c)d < c(a + d),$$

that is,

$$bd + cd < ca + cd,$$

that is,

$$bd < ca,$$

which is the condition that $\frac{c}{d} > \frac{b}{a}$.

In just the same way it may be shown that $\frac{(b + c)}{(a + d)} > \frac{b}{a}$.

3. If we defined $\frac{r}{s} + \frac{t}{u}$ to be $\frac{(r + t)}{(s + u)}$, by the previous problem we would have that the sum of the two fractions would be less than the greater of the two, which would be inconvenient. Also, $\frac{r}{s} = \frac{rc}{sc}$ would result in

$$\frac{rc}{sc} + \frac{t}{u} = \frac{(rc + t)}{(sc + u)}.$$

We shall show that in this case the sum of the two numbers is different from that in the definition - that is, the sum would depend on the fractional form which we happened to use for the number represented by $\frac{r}{s}$. Suppose

$$\frac{(rc + t)}{(sc + u)} = \frac{(r + t)}{(s + u)} .$$

This would be equivalent to

$$(rc + t)(s + u) = (sc + u)(r + t)$$

or

$$rsc + ts + rcu + tu = scr + ur + sct + ut$$

or $$ts + rcu = ur + sct.$$

This can be written

$$ts - sct = ur - rcu$$

or $$ts(1 - c) = ur(1 - c).$$

This can be true only if $c = 1$ or, lacking this, if

$$ts = ur$$

which is equivalent to the equality of the two given fractions

$$\frac{r}{s} \text{ and } \frac{t}{u} .$$

Thus, unless $c = 1$, our definition would give a different form for the sum, assuming that the two given fractions were different.

4. Here it is a matter of checking only. The proofs of these two properties are given in reference 7 pp. 23-26.

5. We now wish to compare the two fractions:

$$\frac{a}{b}, \frac{a + c}{b + c} .$$

If $\frac{a}{b} = 1$, then both fractions are equal to 1. Otherwise, we know from Problem 2 of this section that

$$\frac{a + c}{b + c} \text{ is between } \frac{a}{b} \text{ and } \frac{c}{c} = 1.$$

In other words,

$$\text{if } \frac{a}{b} < 1, \text{ then } \frac{a}{b} < \frac{a + c}{b + c} < 1,$$

$$\text{if } \frac{a}{b} > 1, \text{ then } \frac{a}{b} > \frac{a + c}{b + c} > 1.$$

Thus, adding a positive number, c, to the numerator and denominator of the fraction $\frac{a}{b}$ increases the number represented if $\frac{a}{b} < 1$ and decreases the number represented if $\frac{a}{b} > 1$. These results may also be proved directly; see the solution of Problem 9 of Section 8.2.

Section 5.11

1. The number $\frac{a}{b}$ corresponds to the pair (a,b). The horizontal ray will contain those points for which $b = 0$. But for $b = 0$ there is no number of the form $\frac{a}{b}$.

2. If (a,b) were another way of writing $a + b$, then 4 will correspond to the set of counting numbers a, b whose sum is 4. This set is

$$(0,4), (1,3), (2,2), (3,1), (4,0).$$

This will be a set of five points on a certain line. The number 5 will correspond to those pairs whose sum is 5; there will be six such pairs. For a sum n, there will be n + 1 pairs. In all cases each set of points will be on the same line. All the lines are parallel.

3. If (a,b) represents ab, the points of the lattice which represent 5 would be (1,5) and (5,1). The number 6 would be associated with a set of four pairs:

$$(1,6),\ (2,3),\ (3,2),\ (6,1).$$

The number 7 would have just two pairs associated with it. In general the number of pairs associated with the number of n would be equal to the number of factors of n. These points would in all cases lie on a curve but, except for prime numbers, not on a straight line.

4. Suppose (a,b) were to correspond with a - b. The points of the lattice which would correspond to whole numbers would be those for which b is not greater than a. The pairs corresponding to n would be:

$$(n,0),\ (n+1,1),\ (n+2,2),\ (n+3,3),\ \ldots\ .$$

There would be an infinite number of pairs all on the same straight line.

Section 6.5

1. If $\frac{p}{q}$ is a rational number, we find its decimal expansion by dividing p by q. In the beginning, if p is greater than q, parts of the number p will be "brought down" in the process of long division. But after a certain point, the dividend at each step will be the remainder of the previous step multiplied by ten; and since the rest of the process depends only on the dividend, if the dividend repeats so will the decimal expansion. But there are only q possible different remainders. Hence either the process of division stops or it continues without end, that is, more than q different steps. This means that for the infinite decimal expansion, the remainders must repeat, hence the dividends repeat and therefore in the expansion the same sequence of digits recurs without end. We illustrate this by the following example:

```
        42.153846......
   13/ 548.000000
       52
        28
        26
         20
         13
          70
          65
           50
           39
          110
          104
            60
            52
             80
             78
              20
```

Here the first two remainders are 2 but the repetition does not begin until the eighth step since the remainders in the second and eighth steps are both 2 and in both these cases zeroes are "brought down" in the process. It is the sequence 153846 which is the repeating part of the expansion.

2. In the process of finding the decimal expansion of $\frac{1}{7}$ the first remainder is 3 and the second remainder 2. Since the decimal expansion of $\frac{1}{7}$ is

$$.142857142857\ldots,$$

starting with the second step, the expansion would be

$$.285714285714\ldots.$$

This is the expansion of $\frac{2}{7}$.

3. If we find the decimal expansion of $\frac{1}{9}$ each remainder will be 1; this is because 10 is 1 more than 9. It is this property which makes the sum of the digits have the same remainder when divided by 9 as when the number itself is divided by 9. For 11, the remainders are alternately 10 and 1. Thus, for instance,

$$\begin{aligned} 1367 &= 1000 + 3(100) + 6(10) + 7 \\ &= (990 + 10) + 3(99 + 1) + 6(10) + 7 \cdot 1. \end{aligned}$$

So the remainder when 1367 is divided by 11 is the same as when the following is divided by 11:

$$10 + 3 + 10(6) + 7.$$

For 37 there are three digits in the repeating part. Thus

$$\begin{array}{r} .027027027\ \ldots \\ 37\overline{)\,100000000} \\ \underline{74} \\ 260 \\ \underline{259} \\ 1 \end{array}$$

and the remainders are, in succession: 10, 26, 1. Thus, for instance

$$34578 = 3(10,000) + 4(1,000) + 5(100) + 7(10) + 8$$
$$= 8\cdot 1 + 7(10) + 5(74 + 26) + 4(999 + 1) + 3(9990 + 10)$$

where 74 and 999 are multiples of 37. Thus the remainder when 34578 is divided by 37 is the same as when the following sum is divided by 37:

$$8 \cdot 1 + 7 \cdot 10 + 5 \cdot 26 + 4 \cdot 1 + 3 \cdot 10.$$

4. If a rational number is to have a finite decimal expansion, some power of ten multiplied by the number must be an integer. In other words, if $\frac{p}{q}$ has a finite decimal expansion where 1 is the g.c.f. of p and q, then $(\frac{p}{q})10^n$ must be an integer. Thus

$$p \cdot \frac{10^n}{q}$$

must be an integer. But since p and q have no factors greater than 1 in common, q must be a factor of 10^n. This means that the only prime factors which q can have are 2 and 5. Also, if q has no other prime factors, it will be a factor of some power of 10. For instance, if $q = 2^a \cdot 5^b$ then q is a factor of $10^c = 2^c \cdot 5^c$ where c is the greater of a and b.

5. In the numeral system to the base seven, .1 would mean $\frac{1}{7}$ just as in the decimal system .1 means $\frac{1}{10}$. This is a terminating decimal. In the numeral system to the base seven the expansion of $\frac{1}{5}$ is as follows:

```
   .12541254...
 5/1000000
   5
   20
   13
    40
    34
     30
     26
      1
```

This expansion repeats. In this system, moving the point one space to the left would be equivalent to dividing by seven.

6. Two decimal expansions for $\frac{1}{8}$ are

$$\frac{1}{8} = .125 = .124999\ldots\ .$$

However $\frac{1}{3}$ has only one decimal expansion since at no point in the expansion is there an unending succession of nines or zeros. To see this suppose we change one digit in the expansion of $\frac{1}{3}$. For instance, compare

$$.333\ldots \text{ with } .334000.$$

We have the inequalities:

$$\frac{1}{3} = .333\ \ldots \leq .3334 < .334000 \leq .334\ldots$$

when the remaining digits in the last number can be what you please. This shows that $\frac{1}{3}$ and the last number must differ by at least .0006 and hence cannot be equal. The same argument could be used for .332000 in place of .334000. Without a formal proof it is perhaps clear that the only numbers which can have two decimal expansions are those which from a certain point on have only zeros in their decimal expansions or only nines.

7. Our argument used in finding the fraction representing 5.234234... was based on the fact that this is a number - as we add terms in the expansion we are adding smaller and smaller amounts. On the other hand, for

$x = 1 + 2 + 2^2 + 2^3 + \ldots$ no number is represented by this infinite sum - each time we add a larger and larger amount. All the argument shows is that if x were a number it would have to be -1.

Section 6.6

1. Suppose the population of the city were 10,000. Then 6% of this is 600 and its population at the end of the first year would be 10600. Now 6% of 10600 is 636. Hence the decrease of the second year would be 636 and the population at the end of the second year would be 9,964 which is less than it was in the beginning. It can be seen that for any population the results would be similar since the 6% decrease is computed on a larger amount than is the 6% increase. Had the decrease been first and the increase second, the final population would have been less than that at the beginning, namely, 9,964 just as before.

2. Using 10,000 as the populations of the cities, it can be seen that at the end of two years the population of city A would be 11,200 while that of B would be 11,236, which is more.

3. If the person were really paying interest at the rate of 6% annually or $1\frac{1}{2}$% quarterly, a table of the interest paid for each quarter would be as follows: The first quarter he would be paying interest on the full amount: $1\frac{1}{2}$% of 400 which is $6.00. The second quarter having paid off $100 of the principal as well as the quarter's interest, he would have interest to pay on only $300 which is $4.50. For the third quarter he would be paying interest on only $200, or $3.00, and for the last quarter the interest would be $1.50. Hence the total amount of interest paid under this scheme would be [illegible].00, which is a little more than half of the $24 paid under the other scheme.

To find the actual rate paid under the bank's requirement, let r be the quarterly rate and see that the total interest would be:

$$400r + 300r + 200r + 100r = 1000r.$$

If this is 24, $r = .024$ would be the quarterly rate. This corresponds to an annual rate of 9.6%.

Section 7.4

1. If $b = {}^{-}s$, then $({}^{-}a)({}^{-}s) = as$ and ${}^{-}ab = {}^{-}((a)({}^{-}s)) = {}^{-}({}^{-}as) = as$. Hence both products are equal. Second, write $a = {}^{-}r$ and have

$$({}^{-}a)(b) = rb;\ {}^{-}(ab) = {}^{-}(({}^{-}r)(b)) = {}^{-}({}^{-}rb) = rb.$$

Finally if both a and b are negative we have

$$({}^{-}a)(b) = r({}^{-}s) = {}^{-}(rs);\ {}^{-}(ab) = {}^{-}(({}^{-}r)({}^{-}s)) = {}^{-}(rs).$$

2. In the respective cases we have

i) $({}^{-}a)({}^{-}b) = ({}^{-}a)s = {}^{-}(as) = a({}^{-}s) = ab$

ii) $({}^{-}a)({}^{-}b) = r({}^{-}b) = {}^{-}rb = ({}^{-}r)(b) = ab$

iii) $({}^{-}a)({}^{-}b) = rs = ({}^{-}r)({}^{-}s) = ab.$

3. For the first let $a = 5$, $b = 3$, $c = 1$. Then $(a - b) - c = 2 - 1 = 1$, while $a - (b - c) = 5 - 2 = 3$ and the two are not equal. On the other hand $(a - b) - c = a + {}^{-}b + {}^{-}c$ since if we add c and then b to either side we get a. The same is true of $a - (b + c)$ and $(a - c) - b$.

Section 7.5

If a and b are positive $|a| = a$, $|b| = b$ and $|a| \cdot |b| = ab = |ab|$. If they are both negative, $|a| = {}^{-}a$, $|b| = {}^{-}b$ and $|a| \cdot |b| = ({}^{-}a)({}^{-}b) = ab = |ab|$. If a is negative and b positive, then

$$|a| = {}^{-}a,\ |b| = b,\ |a| \cdot |b| = ({}^{-}a)b = {}^{-}(ab) = |ab|.$$

Section 7.7

1. Here we use the correspondence $(a,b) \leftrightarrow a - b$. Hence if (a,b) corresponds to the integer r, we must have

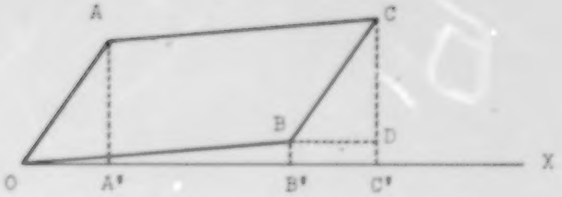
$$a - b = r.$$

Thus the point (a,b) will correspond to r if and only if $a - b = r$. All of these lines will be parallel since

$$a - b = r \text{ and } a - b = s$$

for a pair of values a and b can hold only if $r = s$. Thus none of the lines will have a point in common and since they are all in the same plane they must be parallel.

2. As in the figure, let A be the point (a,b) and B the point (c,d) with O the origin.



Let C be the point $(a + c, b + d)$, A', B' and C' the feet of the perpendiculars from A, B and C upon the x-axis, and D the foot of the perpendicular from B upon $\overleftrightarrow{CC'}$. Then $OA' = a = BD$, $AA' = b = CD$, and the right triangles $AA'O$ and CDB are congruent, showing that $OA = BC$. Similarly it may be shown that $OB = AC$. Hence OACB is a parallelogram and C is the intersection of the line through B parallel to $\overleftrightarrow{OA}$ and the line through A parallel to $\overleftrightarrow{OB}$. This is sometimes thought of as completing the parallelogram starting with the sides $\overline{OA}$ and $\overline{OB}$. If $\overline{OA}$ and $\overline{OB}$ are thought of as two vectors, $\overline{OC}$ is the resultant vector.

3. To verify that

$$(a,b) = (c,d) \text{ if and only if } a + d = c + b$$

is an equivalence relationship, we check the three properties in turn.

i) $(a,b) = (a,b)$ since $a + b = a + b$.

ii) Here $(a,b) = (c,d)$ implies $a + d = c + b$, that is,

$c + b = a + d$, that is, $(c,d) = (a,b)$.

iii) In this case

$(a,b) = (c,d)$ implies $a + d = c + b$.

$(c,d) = (e,f)$ implies $c + f = e + d$.

Thus from the well-defined property of addition

$$(a + d) + e = (c + b) + e,$$

$$a + (e + d) = a + (c + f).$$

Then by the associative and commutative properties

$$a + (c + f) = (c + b) + e,$$

$$(a + f) + c = (b + e) + c = (e + b) + c,$$

and, by the cancellation property for addition,

$$a + f = e + b,$$

which implies $(a,b) = (e,f)$.

Section 7.9

We wish first to show that in a field in which the well-defined property holds for addition, the cancellation property holds for addition. That is, if $a + b = c + b$, then $a = c$. This results from adding ^-b to both sides and using the associative property. For multiplication we similarly start with $ab = cb$, $b \neq 0$ and multiply by the multiplicative inverse $\frac{1}{b}$ of b.

Section 7.11

1. See the answer to the problem of Section 7.9.

2. The first part of the cancellation property for multiplication and inequality is:

   If $ac < bc$ then $a < b$ if c is positive and $a > b$ if c is negative. Now $\frac{1}{c}$ is positive if c is positive and negative if c is negative. Hence, from the well-defined property:

   $$ac < bc \text{ implies } (ac)(\tfrac{1}{c}) < (bc)(\tfrac{1}{c}) \text{ or } a < b, \text{ if } c \text{ is positive.}$$

   If c is negative then $\frac{1}{c}$ is negative and the proof is similar. If the inequalities go in the other direction the proof is the same.

3. P does not form a group because 2, for instance, is in P but $\frac{1}{2}$, its multiplicative inverse is not. The set P', however, does form a group. Neither N nor N' form a group because the closure property is lacking - that is, the product of two negative numbers is not negative. The union of P and N does not form a group since again 2 is in the union but $\frac{1}{2}$ is not. The non-zero rational numbers form a group under multiplication. The set of whole numbers does not since 2 is a whole number but $\frac{1}{2}$ is not.

Section 7.12

1. Property 1 follows from the well-defined property for addition and inequality as follows:

   $$a < b < c \text{ implies } a + d < b + d < c + d$$

   and similarly if the inequalities are all reversed.

2. For the proof of Property 2, one needs also to consider the case $a > b > c$. But the proof is the same - indeed one may merely interchange a and c.

3. See Problem 2 Section 3.2.

Section 7.13

1. The conclusion does not hold, for $2 > -3$ but $2^2 < (-3)^2$.

2. Here we consider two cases. First, if $b > 0$, then the previous problem implies that $a^2 > b^2$ and we have

$$a > b \text{ implies } a \cdot a^2 > a \cdot b^2 > b \cdot b^2 = b^3.$$

If a is positive and b negative or zero, then so are a^3 and b^3 respectively. Finally if both are negative, $a > b$ is equivalent to ${}^{-}a < {}^{-}b$ and by the first case $({}^{-}a)^3 < ({}^{-}b)^3$, or ${}^{-}(a^3) < {}^{-}(b^3)$ which implies $a^3 > b^3$. So in all cases the conclusion is justified.

3. From the triangle inequality we have

$|a - b| + |b| \geq |a|$ which is equivalent to $|a - b| \geq |a| - |b|$

$|b - a| + |a| \geq |b|$ which is equivalent to $|b - a| \geq |b| - |a|$.

But $|a - b| = |b - a|$ and it is greater than or equal to both $|a| - |b|$ and $|b| - |a|$. Hence it is greater than the absolute value of this difference.

4. Since both sides of the equality

$$|a + b| = |a| + |b|$$

are positive it is equivalent to $(a + b)^2 = a^2 + |2ab| + b^2$ that is,

$$a^2 + 2ab + b^2 = a^2 + |2ab| + b^2.$$

This holds if and only if $2ab = |2ab|$, in other words $ab \geq 0$.

5. By Theorem 1, the desired inequality is equivalent to

$$(b + \frac{1}{b})^2 > (1.7)^2.$$

That is:

$$b^2 + 2 + \frac{1}{b^2} > 2.89.$$

But certainly either b^2 or $\frac{1}{b^2}$ is greater than or equal to 1 and hence the left side must be greater than 3 which is greater than 2.89. It is a little harder to show the inequality with 1.7 replaced by 2, the value of the left-hand side when $b = 1$.

Section 8.2

1. Suppose $(\frac{a}{b})$ were a rational number whose square is 3. Then, as in Section 8.1 we would have

$$3b^2 = a^2.$$

Then, as before, 3 will occur as a factor of the left side an odd number of times and of the right side an even number of times, which is impossible.

2. If $\frac{a}{b}$ were a rational number whose cube is 2, then, as before,

$$2b^3 = a^3.$$

The number of times 2 could occur as a factor on the right side would have to be a multiple of 3, that is, one of 0, 3, 6, 9, ... On the left the number of times it could occur is one of

1, 4, 7, 10.

None of the numbers in the first set is equal to a number of the second set and we again have a contradiction.

3. Suppose r is a rational number different from zero, s an irrational number and $rs = t$ a rational number. Then $(\frac{1}{r})t$ would be rational since it is the product of two rational numbers. This would be equal to $(\frac{1}{r})rs = s$ which is irrational. This is a contradiction.

4. Since $\sqrt{2}$ is irrational, the previous problem shows that $(s - r)(\frac{\sqrt{2}}{2})$ must be irrational as well as its sum with r. The number is between s and r since we have added to r something which is less than the difference between s and r.

5. A rational number between $\sqrt{2}$ and $\sqrt{3}$ is 1.415. For the rest, see Section 9.3.

6. First, the decimal for $\frac{1}{11}$ is .090909.... Hence, for this

$$L_1 = .0,\ L_2 = .09,\ L_3 = .090,\ L_4 = .0909,\ L_5 = .09090$$

$$U_1 = .1,\ U_2 = .10,\ U_3 = .091,\ U_4 = .0910,\ U_5 = .09091.$$

Notice that pairs of the U's are equal.

Second, the decimal for $\frac{2}{5}$ is .400000.... Here

$$L_1 = .4,\ L_2 = .40,\ L_3 = .400,\ L_4 = .4000,\ L_5 = .40000$$

$$U_1 = .5,\ U_2 = .41,\ U_3 = .401,\ U_4 = .4001,\ U_5 = .40001.$$

Here all the L's are equal. (We wrote them in different form to emphasize the means of computation of the U's).

Third, using the decimal for π to five places 3.14159

$$L_1 = 3. \quad L_2 = 3.1 \quad L_3 = 3.14 \quad L_4 = 3.141 \quad L_5 = 3.1415$$

$$U_1 = 4. \quad U_2 = 3.2 \quad U_3 = 3.15 \quad U_4 = 3.142 \quad U_5 = 3.1416\ .$$

7. We found above, the two sequences for the decimal .400000.

We now write them for the decimal .39999...

$$L_1 = .3,\ L_2 = .39,\ L_3 = .399,\ L_4 = .3999,\ L_5 = .39999$$

$$U_1 = .4,\ U_2 = .40,\ U_3 = .400,\ U_4 = .4000,\ U_5 = .40000\ .$$

Here all the U's are the same. Notice that the set of U's for the decimal .3999... is the same as the set of L's for the decimal .4000... .

8. Here it is well to recall the result of Problem 5 of Section 5.10. This shows that since $\frac{1}{2}$ is less than 1, the L's form an increasing sequence and, since $\frac{3}{2}$ is greater than 1, the U's form a decreasing sequence. This establishes Properties 1 and 2. To establish the other two we need to show that for every positive integer n, the following difference is positive and that it approaches zero as n becomes larger and larger:

$$U_n - L_n = \frac{3 + n}{2 + n} - \frac{1 + n}{2 + n} = \frac{2}{2 + n} .$$

Thus the difference is positive and as n becomes larger and larger, it becomes smaller and smaller. The number defined by the sequences is 1. Hence we have established our desired results.

9. Here we first need a result analogous to that of Problem 5 of Section 5.10. In this case we start with a fraction, add a positive number to the numerator and twice that number to the denominator. Then the inequality:

$$\frac{c}{d} < \frac{c + r}{d + 2r} ,$$

since all the letters stand for positive numbers, is equivalent to each of the following sequence of inequalities

$$c(d + 2r) < d(c + r)$$

$$2rc < rd$$

$$2c < d$$

$$\frac{c}{d} < \frac{1}{2} .$$

The same results hold if all the inequalities are reversed. Hence, since $\frac{1}{4}$ is less than $\frac{1}{2}$, the sequence of L's is an increasing sequence and since $\frac{3}{4}$ is greater than $\frac{1}{2}$, the sequence of U's is a decreasing sequence. Thus the first two properties desired are true. For the rest we compute:

$$U_n - L_n = \frac{3 + n}{4 + 2n} - \frac{1 + n}{4 + 2n} = \frac{2}{4 + 2n} = \frac{1}{2 + n} .$$

This is positive and approaches zero as n becomes larger and larger. The number defined by the sequence is $\frac{1}{2}$. See also the solution of Problem 5 of Section 5.10.

Section 9.2

1. Let A and B stand for two equations and S and T their solution sets respectively. Then A and B are equivalent if and only if $S = T$. Since equality of sets is an equivalence relationship, so must be equivalence of equations.

2. Here the two equations are equivalent except for those values of x for which $x - 1 = 0$, that is, except for $x = 1$. Hence the only possible solutions of the given equation are $x = 3$, the solution of the second equation, and $x = 1$. In this case $x = 1$ is a solution of the first equation. This equation may also be solved by use of the distributive property as follows:

$$(x - 2)(x - 1) = x - 1$$

is equivalent to

$$(x - 2)(x - 1) - (x - 1) = 0.$$

By the distributive property, this is equivalent to

$$(x - 2 - 1)(x - 1) = 0$$

$$(x - 3)(x - 1) = 0$$

which has the solutions $x = 3$ and $x = 1$.

3. Here to solve the given equation we multiply first by $(x - 1)(x - 2)$ as in the example in the section previously. This yields

$$x - 2 + 1 = (x - 1)(x - 2)$$

$$x - 1 = (x - 1)(x - 2). \tag{2}$$

This equation is equivalent to the given one except for those value of x for which the multiplier is 0, that is, $x = 1$ and $x = 2$. Now equation (2) is the same as the one considered in Problem 2 and we found that its solutions are $x = 1$ and $x = 3$. Thus the solutions of the equation of this problem are $x = 3$ and perhaps $x = 1$, and $x = 2$. For the two last values of x, denominators are zero. Hence the only solution of the given equation is $x = 3$.

Section 9.3

1. Exactly the same proof can be used here as in the proof of the lemma of this section.

2. Since D contains a number $\frac{1}{2^k}$ as small as you please, we can use the same argument to prove the result required in this problem as for the density of rational numbers.

3. Here we wish to choose q, an element of D, so that r satisfies the conditions imposed with $r = a - bq$. Thus we want

i) $$a - bq \geq 0 \quad \text{and} \quad \text{ii)} \quad a - bq < c.$$

The first inequality is equivalent to

$$a \geq bq \quad \text{and} \quad \frac{a}{b} \geq q.$$

The second inequality is equivalent to

$$a - c < bq \quad \text{and} \quad \frac{a - c}{b} < q.$$

Thus, since

$$\frac{a - c}{b} < \frac{a}{b}$$

for a, b and c positive, we want to find a number of D which is between $\frac{(a - c)}{b}$ and $\frac{a}{b}$. Since from the previous problem we know that there is a number of D between any two rational numbers, we know that there is a number of D which is between the two special rational numbers $\frac{(a - c)}{b}$ and $\frac{a}{b}$. This completes the proof.

Section 9.7

1. It is probably easiest to consider this problem by cases.

Case I. If $aa' \neq 0$, then the given pair: $f = 0$, $f' = 0$ is equivalent to each of the following sequence of pairs:

$$a'f = 0,\ af' = 0;$$

$$a'f - af' = 0,\ f = 0$$

as was shown in the previous section. The last pair above is

$$a'(ax + by + c) - a(a'x + b'y + c') = 0,\ ax + by + c = 0$$

or

$$(a'b - b'a)y + a'c - ac' = 0,\ ax + by + c = 0. \qquad (1)$$

Thus, if $a'b - b'a \neq 0$, the first of these equations has one and only one solution y; and, using this value of y in the second equation we see that there is exactly one value of x. Since this pair is equivalent to the given pair, we have shown that if $a'b - b'a \neq 0$, the given pair has exactly one solution.

On the other hand, if $a'b - b'a = 0$, the first equation of (1) has no solution if $a'c - c'a \neq 0$, while if $a'c - c'a = 0$, any value of y is a solution, which in turn yields values of x by means of the second equation of (1). Thus we have proved the theorem for Case I.

Case II. If $bb' \neq 0$, then we may use the same proof as in Case I with b and a, x and y interchanged. Notice that this interchange does not change the condition $a'b - b'a = 0$.

Case III. If $a = b = 0$ then either $c \neq 0$ when there is no solution or $c = 0$ when the pair of equations is in reality only one equation which either has infinitely many solutions or none. Also for this case, $a'b - b'a = 0$. The case $a' = b' = 0$ is similar.

Case IV. If $a \neq 0 = b$, $a' = 0 \neq b'$ the given pair of equations reduces to:

$$ax + c = 0,\ b'y + c' = 0.$$

Here the unique solution is $x = -\frac{c}{a}$, $y = -\frac{c'}{b'}$ and $ab' - ba' = ab' \neq 0$.

Case V. If $a = 0 \neq b$ and $a' \neq 0 = b'$, the proof is the same as in Case IV. We have considered all possible cases and the result is established.

2. The answer to the question has been given in the discussion of Problem 1, i.e., if $ab' - a'b = 0$, then there is either no solution or an infinite number of solutions. Geometrically, this means that if $ab' - a'b \neq 0$, the lines intersect, while if $ab' - a'b = 0$ they either are parallel or coincide.

3. Here we have two pairs of equations to consider:

$$f = 0, \ g = 0$$

and

$$rf + sg = 0, \ tf + ug = 0.$$

We know that any solution of the former pair is a solution of the latter. Thus, the pairs will be equivalent if the last two equations hold only for $f = 0$ and $g = 0$. But we showed in the solution of Problem 1 that the pair $rf + sg = 0$, $tf + ug = 0$ has a single solution ($f = 0$, $g = 0$) if and only if $ru - ts \neq 0$. Thus the condition that the two pairs be equivalent is

$$ru - ts \neq 0.$$

Section 9.8

1. Here let x be the amount of the 80% Java mixture used and y the amount of the 50% Java mixture used. Then the combined amount is to be five pounds which gives us

$$x + y = 5.$$

The amount of Java in the first is $.8x$ and in the second $.5y$. Thus

$$.8x + .5y = (.6)5 = 3.$$

So we wish to solve the pair of equations

$$x + y = 5$$

$$.8x + .5y = 3.$$

To do this multiply the first equation by .8 to have

$$.8x + .8y = 4.$$

Subtracting, we have

$$.3y = 1,$$

$$y = \frac{10}{3}.$$

Thus $x = \frac{5}{3}$ and this is the solution.

2. To justify the geometric method use Figure 2 of Section 9.8. Let A be the point (5,3) and recall that P is the point where the line through A parallel to J = .5W cuts the line J = .8W. Let C be the foot of the perpendicular from P on the W-axis, call x the first coordinate of P (and of C) and B the point where the horizontal line through P cuts the vertical line through A. Since x is the first coordinate of P, the distance CP is the amount of Java coffee in x pounds of the 80% mixture. The distance BA is the amount of Java coffee in PB pounds of the 50% mixture, since the line $\overleftrightarrow{PA}$ is parallel to the line J = .5W. The sum of BA and PC, namely 3, will be the amounts of Java coffee in x pounds of the 80% mixture plus PB pounds of the 50% mixture. But PB + x = 5. Thus x pounds of the 80% mixture plus PB pounds of the 50% mixture give us 5 pounds of the 60% mixture. That is, we must use x pounds of the 80% mixture plus 5 - x pounds of the [illegible]0% mixture to get 5 pounds of the 60% mixture.

3. Here we wish conditions on A and B in

$$g + 2c = A$$

$$75g + 50c = B$$

so that the values of g and c in the common solution will both be positive. Multiply the first equation by 75 and get

$$75g + 150c = 75A$$

$$75g + 50c = B$$

and $$100c = 75A - B.$$

Hence $$c > 0 \text{ is equivalent to } 75A > B.$$

On the other hand if we multiply the first equation by 25 we have

$$25g + 50c = 25A,$$

$$75g + 50c = B,$$

which gives

$$50g = B - 25A$$

and the condition that g be positive is $B > 25A$.

Thus the condition that both g and c be positive is

$$25A < B < 75A,$$

in other words, B must be between 25A and 75A.

Appendix I, Section 3

1. A and B are sets of points. Their intersection can be the null set, in which case they are parallel, being in the same plane. Their intersection can be a single point. The only other possibility is $A = B$.

2. Suppose A and B are two null sets. Since the null set is contained in every set, we have $A \subset B$ and $B \subset A$. This implies $A = B$.

3. $A \cap B$ means the set of elements in both A and B. Hence every element of the intersection is in A and also in B.

4. Since $A \cup B$ contains all elements in either A or B or both, every element of A is contained in this union and similarly for B.

5. Since every element of the intersection of two sets is in both, $A \cap B = B$ implies that B is contained in A.

6. If $A \cup B = B$, then every element of A must be contained in B. Hence A is a subset of B.

7. First: $A \cup B$ is the set of elements in A or B or both. This is the same as the set of elements in B or A or both. Hence $A \cup B = B \cup A$. Similarly $A \cap B = B \cap A$ because any element in both A and B is in both B and A. For the associative property $A \cup (B \cup C)$ is the set of elements in one or more of A, B and C. Finally we want to show

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

If an element is in B and C, it is in the set depicted on the left side; it is also in $A \cup B$ and $A \cup C$ and hence in the set of the left side. If an element is in A, it is in the set of the left side and in both $A \cup B$ and $A \cup C$ on the right. Thus every element in the set on the left is in the set on the right. Also any element on the right must be in A or in the intersection of B and C. Thus the proof is complete.

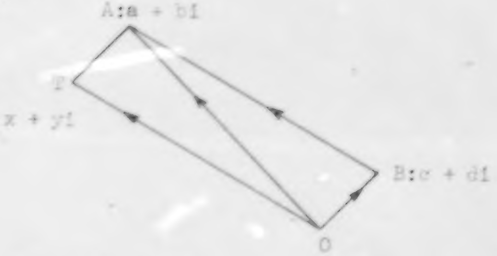
Appendix III, Section [illegible]

1. Suppose $a + bi$ is a real number. Then $b = 0$. The absolute value considered as a complex number is $\sqrt{a^2}$. This is the same as the absolute value of a considered as a real number since it is equal to a if a is positive and to ^-a if a is negative. (Recall that $\sqrt{b}$ means the positive square root of b if b is a positive number.)

2. Let the point A have the coordinate $a + bi$ and B the coordinate $c + di$. We first show that the distance AB is equal to the absolute value of the number:

$$a - c + (b - d)i.$$

Let O denote the origin of coordinates and T the point where the line through O parallel to $\overleftrightarrow{AB}$ intersects the line through A parallel to $\overleftrightarrow{OB}$, as in the figure:



Then the segments $\overline{TO}$ and $\overline{AB}$ have the same length since they are opposite sides of a parallelogram. But the vector $\underline{OA}$ is the resultant of the vectors $\underline{OT}$ and $\underline{OB}$, that is, if $x + yi$ is the coordinate of T, we have:

$$(x + yi) + (c + di) = a + bi.$$

Hence,

$$x + yi = a + bi - (c + di) = (a - c) + (b - d)i.$$

Thus we have shown that the distance OT, and hence AB, is equal to the absolute value of $(a - c) + (b - d)i$. Thus*

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*This formula was derived in a different way in Exercise 2 of Section 7.7.

$$AB = \sqrt{(a - c)^2 + (b - d)^2} \text{ .}$$

To complete the solution of the problem, notice that by the Pythagorean Theorem, OAB is a right triangle with the right angle at O, if and only if,

$$(OA)^2 + (OB)^2 = (AB)^2 .$$

Thus

$$(a^2 + b^2) + (c^2 + d^2) = (a - c)^2 + (b - d)^2$$

$$= a^2 - 2ac + c^2 + b^2 - 2bd + d^2 .$$

This equality holds if and only if:

$$ac + bd = 0.$$

Appendix IV, Section 7

Here, using the notation of Section 2, B to the left of line $\overleftrightarrow{AD}$ implies: $b_1 < a_1$. This, coupled with the assumption that B is above the horizontal line through A, gives

$$B'B = b_2 - a_2 \text{ and } AB' = a_1 - b_1 .$$

Thus

$$-s(A,B) = \frac{b_2 - a_2}{a_1 - b_1} = \frac{B'B}{AB'} \text{ .}$$

First, if C is above the horizontal line through A, it must be to the left of $\overleftrightarrow{AD}$ and

$$C'C = c_2 - a_2 , \; AC' = a_1 - c_1 .$$

Thus

$$-s(A,C) = \frac{c_2 - a_2}{a_1 - c_1} = \frac{C'C}{AC'} \text{ .}$$

Then the equality of the ratios $\frac{B'B}{AB'}$ and $\frac{C'C}{AC'}$ implies

$$-s(A,B) = -s(A,C), \text{ that is, } s(A,B) = s(A,C).$$

Second, if C is below the horizontal line through A, it must be to the right of $\overleftrightarrow{AD}$ and

$$C'C = a_2 - c_2, \quad AC' = c_1 - a_1.$$

These are both the negatives of the values of the previous case and hence the ratio is the same and the rest of the proof follows.

Answers and Partial Answers to Selected Exercises

Section 2.7

2. $7^2, 4^3, 3^4, 2^7$

3. c) ete01

4. The multiples of 5 are: 5, 12, 17, 24, 31, 36, 43.

   The multiples of 7 are: 7, 16, 25, 34, 43, 52, 61.

5. b. 11011111001 d. .001001001...

6. Parts c and d would indicate that $\frac{1}{1111}$ would have the expansion .000100010001... and similarly for other denominators consisting of all 1's.

7. Six dozen, in the numeral system to the base seven, is written 132.

Section 3.3

2. c) $x = 2$

4. The intersection of the two sets may not be the null set.

5. One can add the first and the last, then the second and the next to the last, the third ahd the third from the last, etc.

6. $9 + 18 + 27 + 36 + \ldots + 90 + 99 = 594$

7. The union of sets A and B would be the set of people who like ice cream, chocolate or both and the intersection is the set of people who like both. The formula would then affirm that the number of people who like ice cream plus the number who like chocolate is equal to the number of people who like at least one of the two plus the number who like both.

9. Property 1 holds because if a and b are both odd or both even the relationship is an ordinary inequality which has Property 1. On the other hand if one of a, b is odd and the other even, the relation R holds in only one order.

For Property 2, notice that if a, b, c are all odd we have ordinary inequalities and hence a R b, b R c implies a R c. Suppose this is not the case and first consider a even. Then, for a R b to hold, b must be even and less than or equal to a; similarly for b R c to hold, c must be even and less than or equal to b. Second, suppose a is odd and b even; then for b R c to hold, c must be even also and a R c is true. Third, suppose a is odd, b odd and c even; then a R c. We have considered all cases.

Section 3.6

1. a) 8 - (5 - 3) = 6

2. Eleven different numbers: 1, 3, 5, 7, 9, 13, 15, 17, 25, 29, 31.

4. b) Four different numbers: 2, 4, 5, 20.

5. The properties used are the distributive property, the commutative property of multiplication and the associative property of addition.

6. Since, from Exercise 5, $(x + 1)^2 - x^2 = 2x + 1$ we see that the difference between two successive squares: x^2 and $(x + 1)^2$ is one more than double the first number, e.g., 19 is $2 \cdot 9 + 1$ and the difference between 10^2 and 11^2 is $2 \cdot 10 + 1 = 21$.

7. Looking at the table for Exercise 6, we see that since 4 - 1 = 3, 1 + 3 = 4. Also 1 + 3 + 5 = 9. In general if we add 1 to the sum of the numbers in the second row of the table stopping at some number, then this sum will be equal to the next number on the line above.

8. Here the differences could be found by calculating

$$(x + 2)^2 - x^2 .$$

10. Note that $73 \cdot 137 = 10,001$.

Section 3.8

2. When $*$ stands for addition or multiplication, the equality follows from the commutative properties for addition and multiplication.

   If $*$ stands for subtraction, we have on the left

   $$a - (b - c) = a - b + c = a + c - b.$$

   On the right we have

   $$c - (b - a) = c - b + a = a + c - b,$$

   and the two sides are equal.

3. First notice that division by 1 either inside or outside of a parenthesis does not alter the result. Hence we need consider only

   $$16 \div 8 \div 4 \div 2.$$

   Since not only is the above symbol ambiguous but also $16 \div 8 \div 4$ and $8 \div 4 \div 2$, there must be a pair of parentheses around two successive numbers and there must be another pair. Suppose our final symbol contains $(4 \div 2)$. Then there are two possibilities:

   a) $16 \div [8 \div (4 \div 2)] = 16 \div 4 = 4$

   b) $(16 \div 8) \div (4 \div 2) = 2 \div 2 = 1.$

   If the final symbol contains $(8 \div 4)$, there are two possibilities:

   c) $[16 \div (8 \div 4)] \div 2 = 8 \div 2 = 4$

   d) $16 \div [(8 \div 4) \div 2] = 16 \div 1 = 16 .$

   In conclusion, if the final symbol contains $(16 \div 8)$, then, in addition to b) above we have:

   e) $[(16 \div 8) \div 4] \div 2 = (\frac{1}{2}) \div 2 = \frac{1}{4}.$

   Thus we have four different possible answers: $16, 4, 1, \frac{1}{4}$.

Section 3.9

1. Using the second suggestion we have: $a < b$ implies $ac < bc$ from the well-defined property and $cb < db$ from the same property. These two inequalities, using the transitive property, imply $ac < bd$.

2. The same method may be used here as in Exercise 1.

3. Since $a < b$, then $a + r = b$ for some natural number r. Then $a + c > b + d$ may be written $a + c > a + r + d$ which implies

$$c > r + d$$

and hence $c > d$.

6. For the first we must have $a \geq (b - c)$ and $b \geq c$. For the second $a \geq b$ and $a - b \geq c$. Since $a \geq b$ implies $a \geq b - c$, we have three conditions

$$a \geq b \geq c \quad \text{and} \quad a - b \geq c,$$

for natural numbers.

Section 4.2

2. Suppose the factors of N are: $1, a, b, c, N$ and those of M are $1, d, e, M$. Then factors of MN are:

$$1, a, b, c, N;\ d, da, db, dc, dN;$$
$$e, ea, eb, ec, eN;\ M, Ma, Mb, Mc, MN.$$

If 1 is the only common factor of M and N, then different letters stand for different numbers in the display above and similarly for the products. Hence the number of factors of MN is the product of the numbers of factors in M and N, in this case $5 \cdot 4 = 20$.

3. If b is a factor of c, then $bx = c$ for some integer x. Continuing along this line, the proof may be completed.

5. We know that $b = cq + r$ where $0 \leq r < c$, that is,

$$b - cq = r.$$

If $r < \frac{c}{2}$ we see that cq differs from b by less than $\frac{c}{2}$. If $r > \frac{c}{2}$, the equation is equivalent to

$$b - cq - c = r - c, \text{ or}$$

$$(q + 1)c - b = c - r .$$

But $r > \frac{c}{2}$ implies $c - r < \frac{c}{2}$ and we see that $(q + 1)c$ is a multiple of c which differs from b by less than $\frac{c}{2}$. All this is probably easier to see graphically, since if b is between two successive multiples of c, it must either be at the midpoint of the segment connecting them or be nearer to one end than the other.

7. If $a + b$ and $a - b$ are divisible by 73, their sum, 2a, and their difference, 2b, must also be divisible by 73. Since 73 is odd, this shows that 73 must be a factor of both a and b. The conclusion would be the same for any odd number in place of 73.

8. If we use the factorization $15 = 15 \cdot 1$ we have

$$x + y = 15$$

$$x - y = 1 .$$

Adding gives $2x = 16$ or $x = 8$. This implies $y = 7$. A similar procedure gives solutions for the factorization $5 \cdot 3$. There are no solutions for $n = 22$, for the same procedure would give 2x or 2y an odd number which is impossible for whole numbers x or y.

9. For the remainders 0, 1, 2, respectively, the number being divided by 3 must be of the form: 3n, 3n + 1, or 3n + 2. So, if x is not divisible by 3 it has one of the forms

$$3n + 1, \ 3n + 2 .$$

In the first case $x - 1 = 3n$, which is a multiple of 3, and in the second case $x + 1 = 3n + 3$ which is a multiple of 3. Thus, since either $x - 1$ or $x + 1$ is divisible by 3, their product is divisible by 3. Their product is $x^2 - 1$.

Section 4.3

3. Since the remainder when n is divided by 3 is one of 0, 1, 2, n may be written in one of the forms: $3k$, $3k + 1$, $3k + 2$. In the first case n is divisible by 3, in the second case $n + 2$ is divisible by 3 and in the third $n + 1$ is divisible by 3. Hence one of the three numbers n, $n + 1$, $n + 2$ is divisible by 3. If they are all prime numbers, one of them therefore must be 3. Then form the sets:

$n + 2 = 3$, $n + 1 = 2$, $n = 1$ and 1 is not a prime,

$n + 1 = 3$, $n + 2 = 4$, $n = 2$ and 4 is not a prime,

$n = 3$, $n + 1 = 4$, $n + 2 = 5$ and 4 is not a prime.

Thus no such set can consist of prime numbers alone.

4. Note that, for instance, 3 and 5 are prime numbers which differ by 2.

6. The only one of these which is a prime is 311.

Section 4.4

3. The g.c.f. is 11.

9. Since 11 is the g.c.f. of 17, 325 and 407, 11 divides the left side of the equation $17{,}325x + 407y = c$ and hence must divide the right side, if x and y are integers. Thus, on dividing by 11, we have

$$1575x + 37y = \frac{c}{11}.$$

We know from Theorem 4 that this is solvable if $\frac{c}{11} = 1$, that is, $c = 11$. Thus this value of c is the least for which there is a solution.

To find a solution, we perform the euclidean algorithm for the pair 1575, 37, as follows:

$$1575 = 42 \cdot 37 + 21$$

$$37 = 1 \cdot 21 + 16$$

$$21 = 1 \cdot 16 + 5$$

$$16 = 3 \cdot 5 + 1.$$

Then, starting with the last equation we work upward. The last equation may be written:

(1) $$1 = 16 - 3 \cdot 5.$$

From the next to the last equation we have $5 = 21 - 1 \cdot 16$. Replacing 5 by this in (1) we get

$$1 = 16 - 3(21 - 16) = (1 + 3 \cdot 1) \cdot 16 - 3 \cdot 21$$

(2) $$1 = 4 \cdot 16 - 3 \cdot 21.$$

The second equation in the euclidean algorithm gives $16 = 37 - 21$. Putting this in for 16 in (2) we get

(3) $$1 = 4(37 - 21) - 3 \cdot 21 = 4 \cdot 37 - 7 \cdot 21.$$

Finally, using the first equation of the euclidean algorithm we find: $21 = 1575 - 42 \cdot 37$. Replacing 21 by this in equation (3) we get (after a little pencil work):

$$1 = 298 \cdot 37 - 7 \cdot 1575,$$

which can be written:

$$1 = 1575(-7) - 37(-298).$$

Multiplying through by 11 gives us as a solution of

$$11 = 17{,}325x - 407y$$

$x = -7$ and $y = -298$.

4. Suppose d is a factor of bc and that $d = xy$ where x is a factor of b and y of c. Let g be the g.c.f. of d and b. Since x is a factor of both b and d, it is a factor of g and we have $g = kx$ for some natural number k.

First we shall show that $k = 1$. Now

$$\frac{d}{g} = \frac{d}{xk} = \frac{d}{x} \cdot \frac{1}{k} = \frac{y}{k}$$

and hence k is a factor of y. But k is a factor of g and hence of b, while y is a factor of c. Thus 1 the g.c.f. of b and c implies that the only factor of b which divides y is 1. This shows that $k = 1$ and $g = x$. Thus the conditions given determine x.

Then d a factor of bc implies $bc = dh$ for some natural number h. Then, dividing by x, we get:

$$\frac{b}{x} c = \frac{d}{x} h = gh.$$

Since 1 is the g.c.f. of $\frac{b}{x}$ and y, Theorem 3 implies that y is a factor of c. This completes the proof because we have shown that x is determined by being the g.c.f. of b and d, and y, determined by $d = xy$, is a factor of c.

5. Both questions can be answered in the affirmative. For instance, if $b = 15$ and $c = 63$, the number 35 can be expressed in only one way as a product of a factor of b and one of c. But the number 21 can be written $1 \cdot 21$ or $3 \cdot 7$ where in each case the first member of the product is a factor of 15 and the second member a factor of 63.

6. Using the symbolism of Exercise 7, we have $g = (a,b)$ and $h = (g,c)$ and wish to show that h is the g.c.f. of a, b and c. Now since h divides g which divides a and b, we know that h divides not only c but also a and b. So h must be a factor of the g.c.f. of a, b and c. On the other hand, any common factor of a, b and c must divide g and c and hence h. So $\underline{h}$ is the g.c.f. of a, b and c.

7. Both sides represent the g.c.f. of a, b and c, by the results of the previous exercise.

Section 4.5

1. In Exercise 2 of Section 4.4 it was determined that 11 is the g.c.f. of 17,325 and 407. Thus, from Problem 2 of this section, the l.c.m. of 17,325 and 407 is their product divided by 11. Thus, since $\frac{407}{11} = 37$ and $\frac{17,325}{11} = 1575$, the sum of the two fractions of this exercise is

$$\frac{79 \cdot 37}{17,325 \cdot 37} + \frac{13 \cdot 1575}{407 \cdot 1575}$$

and the denominators are equal to 641,025. Thus the numerator is

$$79 \cdot 37 + 13 \cdot 1575 = 23,398.$$

The fraction $\frac{23,398}{841,025}$ is in simplest form, incidentally, since 11 is not a factor of the numerator.

2. One could find the l.c.m. without factoring, using the results of Exercise 5 below, but it is simplest to note that 185 is $5 \cdot 37$. Since 5 is a factor of 17,325 and 37 of 407, we see that the l.c.m. of all three is the same as the l.c.m. of the first two.

3, 4. The same methods may be used here as in Exercise 6 of the previous section.

5. Here use Problem 7 of this section.

6. The least common divisor of a pair of natural numbers is 1. There is no greatest common multiple for if c is a common multiple, $2c$ is also a common multiple - in fact any multiple of c is a common multiple.

Section 4.6

1. Since 1000 is divisible by 8, and any natural number can be written in the form $1000n + c$, where c is a number less than 1000 which is represented by the last three digits of the given number, it follows that a number is divisible by 8 if and only if the number consisting of its last three digits is divisible by 8. For example, 159,352 is divisible by 8 because 352 is. To test for divisibility by 72 we merely test for divisibility by 8 and by 9.

3. If the sum of the digits is even, the number is divisible by 2, since 11 is 1 more than a multiple of 2.

4. If the base of the numeral system is even, one can test for divisibility by two by looking at the last digit. If the base is odd, one adds the digits.

Section 5.5

1. Here $\frac{a}{b} = \frac{c}{d}$ and $\frac{r}{s} = \frac{t}{u}$ imply ad = bc and ru = st. Then $(\frac{a}{b})(\frac{r}{s}) = \frac{ar}{bs}$ and $(\frac{c}{d})(\frac{t}{u}) = \frac{ct}{du}$. These two fractions will be equal if and only if ardu is bsct. But this follows from ad = bc and ru = st.

2. See the answer to Exercise 1 of Section 5.9.

Section 5.6

1. In the first case $\frac{(\frac{x}{y})}{y}$ is equal to $\frac{x}{y^2}$ and in the second case $\frac{x}{(\frac{y}{y})}$ is equal to $x(\frac{y}{y}) = \frac{xy}{y}$. These are not in general equal.

2. Compare Exercise 2, Section 5.2.

Section 5.7

1. Suppose p were a prime factor of both the numerator and denominator of the fraction $\frac{(ad + bc)}{bd}$. First, it might divide b. In that case to divide the numerator it would have to divide ad + bc and hence ad. But p is a factor of b which has no factors in common with a; hence p divides d. But this is impossible since b and d have 1 as their g.c.f. Hence our assumption that p divides b and ad + bc is false. Similarly we show that p cannot divide d and the numerator. This establishes our result.

2. For the sum of fractions $\frac{a}{b}$ and $\frac{c}{d}$ to be an integer, we would have:

$$\frac{(ad + bc)}{bd} = x,$$

with x an integer, that is,

$$ad + bc = bdx.$$

This shows that b is a factor of ad and, since 1 is the g.c.f. of a and b, b is a factor of d. Similarly we can show that d is a factor of b. This implies that b and d are equal.

Section 5.10

3. Assume $a > b$. Then we may divide both sides by ab and get

$$\frac{a}{ab} > \frac{b}{ab}, \text{ that is } \frac{1}{b} > \frac{1}{a}.$$

4. The given equation is equivalent to: $ax - by = 1$.

5. They can ring $\frac{1}{15}$ -th of an hour apart as may be seen by using the equation in the previous exercise with $b = 3$ and $a = 5$.

Section 6.4

1. a) $327{,}000 \cdot 10^{-4}$

2. a) The approximate relative ratio of error is

$$\frac{.0001}{32} = \frac{1}{3200}\,\% .$$

3. c) The greatest possible error is .0001 feet.

5. The approximate possible percentage of error in the volume of a cube is 3% when the percentage of error in the measurement of a side is 1%.

Section 6.5

1. e) $1000 \cdot 10^2 = 10^5 = 100{,}000$

   f) [illegible],400,000,000

4. Here $\frac{1}{13} = .076923076923\ldots$ with 076923 the repeating part. On the other hand $\frac{2}{13} = .153846153846\ldots$ with 153846. Also $\frac{3}{13} = .230769230769\ldots$ where the repeating part is a cyclic permutation of that for $\frac{1}{13}$. Half the time the repeating part will be related in this way to that for $\frac{1}{13}$ and the other half to that for $\frac{2}{13}$.

Section 6.6

2. The percentages of increase in successive years are:

6%, 5.7%, 5.4%, 5.1%, 4.8% .

3. The percentages of decrease in successive years are:

6%, 6.4%, 6.8%, 7.3%, 7.9% .

4. No, except for the first year. For Exercise 2 the percentages decrease since the numerical increase is the same while the population grows. For Exercise 3 it is just the opposite.

5. For Exercise 2, the total increase over five years is 300 which is a 30% increase. For Exercise 3, the total decrease is 300, that is a 30% decrease. These are the same because they are based on the same population, 1000, at the beginning. In the previous exercises the changes are based on a changing population. Note that to divide 300 by 5 and say that 6% is the average increase or decrease gives a figure which is not in any usual sense the average of the yearly increases (or decreases) since it is greater (or less) than all but the first. Here we use another kind of average, the geometrical mean. (See reference 4, Chap. 7, Sec. 10).

7. Let P stand for the present population. Then $\frac{1}{2}$ P will represent the number of persons under 16 years of age. At the end of 16 years these and their children will amount to a population of $\frac{1}{2}P + 2(\frac{1}{2}P) = (\frac{3}{2})P$. At the end of 16 years the population of the other half will become $(\frac{6}{5})\frac{P}{2}$ The sum of these two amounts is $\frac{21}{10}P$. This means that the population will more than double in sixteen years.

Section 7.4

5. They cannot all be positive nor can exactly two be negative. One or three may be negative.

7. The numbers 5 and $(\frac{1}{5})^{-1}$ are equal, -5 is equal to none of the others. All the rest are equal.

8. $R5 = \frac{1}{5}$, $S5 = -5$, $R(S5) = R(-5) = -\frac{1}{5}$, $S(R5) = S(\frac{1}{5}) = -\frac{1}{5} = R(S5)$, $R(R5) = R(\frac{1}{5}) = 5$, $S(S5) = S(-5) = 5$. The interested reader might like to show that these two operations generate a group.

Section 7.5

4. If $|b| > |c|$ and $b < c$ then b must be negative.

6. Using the results from Exercise 5, we see that $CB = \frac{DC}{3}$. Hence there are two possibilities: 1. B is between C and D and one-third of the way from C to D; 2. C is between D and B with CD three times BC.

Section 7.7

2. The formula follows from the Pythagorean Theorem since A, B and C are the vertices of a right triangle with the right angle at B. The Pythagorean Theorem tells us that:

$$(AC)^2 = (AB)^2 + (BC)^2 .$$

See also the answer to Problem 2 in Appendix III.

3. $\sqrt{37}$

4. Under the conditions given AB = BC.

5. Since (a,b) corresponds to $a - b$, (a,b) will correspond to 1 when $a - b = 1$. Thus the number pair $(b + 1,b)$ corresponds to 1, or, changing letters, $(y + 1,y)$ corresponds to 1. Then

$$\begin{aligned}(a,b)(y + 1,y) &= (a(y + 1) + by,\ b(y + 1) + ay)\\ &= (a + ay + by,\ b + by + ay) = (a,b),\end{aligned}$$

the last equality following from the fact that:

$$a + ay + by + b = b + by + ay + a .$$

Section 7.12

1. If the point corresponding to c is between the points corresponding to the points b and d, then one of two things may happen:

$$b < c < d \quad \text{or} \quad d < c < b.$$

In the former case, $|b - c| = c - b$, $|c - d| = d - c$ and the sum:

$$|b - c| + |c - d| = c - b + d - c = d - b$$

which is equal to $|b - d|$. The other case goes similarly.

2. Let B, C and D be the points corresponding to the numbers b, c and d respectively. Recall that $BC = |b - c|$, etc. Thus:

$$|b - c| + |c - d| = |b - d| \text{ can be written } BC + CD = BD.$$

We showed in Exercise 1 that if C is between B and D, then $BC + CD = BD$. In this exercise we are given $BC + CD = BD$ and asked to show that C is between B and D. This can be done by supposing that C is not between B and D and arriving at a contradiction.

Now we know that one of B, C, D is between the other two by Section 3.2. Suppose D is between B and C. Then by the previous exercise with the letters changed:

$$BD + DC = BC.$$

But we know by hypothesis that $BC + CD = BD$. Thus, using the previous equation we have:

$$BC + CD = BD + DC + CD = BD.$$

This tells us that $DC + CD = 0$. But $DC = CD$ since the distance between D and C is the same as the distance between C and D. Thus we have shown that if D is between B and C and $BD + DC = BC$, then $CD = 0$. This is false and hence D cannot be between B and C. Similarly we can show that B is not between D and C. The only possibility left is that C is between B and D.

5. For this note that the three expressions represent BC, BD, CD respectively and that, by Section 3.2, one point must be between the other two. Then use the results of the previous exercises.

Section 7.13

1. Our results follow directly from Problem 4 if we write the given equality as follows:

$$|a + (-b)| = |a| + |-b|$$

if and only if $a(-b) \geq 0$, that is, $ab \leq 0$.

2. Yes.

3. Since a + c is positive, the given inequality is equivalent to

$$(a + c)^2 + a^2 \geq 2a(a + c) = 2a^2 + 2ac,$$

that is,

$$2a^2 + 2ac + c^2 \geq 2a^2 + 2ac.$$

This, in turn is equivalent to

$$c^2 \geq 0$$

which is true, and the equality holds only if $c = 0$.

4. The given inequality follows from the result of the previous exercise by taking b or -b equal to a + c. If we let $a = 1$, this gives us an improvement on the result of Problem 5, in which 1.7 is replaced by 2. This is the maximum number we can have for the inequality to hold, since the equality actually holds when $b = 1$.

Section 8.2

1. Yes.

2. The process for base seven is essentially the same as that for the decimal system. For example, ten is 13 in the numeral system to the base seven, and the division in this numeral system will be:

$$\begin{array}{r} .0462\ldots \\ 13\overline{)1.000000000} \\ \underline{55} \\ 120 \\ \underline{114} \\ 30 \\ \underline{26} \\ 10 \end{array}$$

and the sequence of digits .0462 repeats in the numeral system to the base seven. Thus this corresponds to one-tenth in the decimal system. Similarly to find one one-hundredth, we note that one hundred is 202 in the system to the base seven and the representation of one one-hundredth in this system is

$$.003300330033\ldots$$

where again it happens that the repeating part contains four digits.

4. From the definition of t_n, we have

$$t_n - t_{n-1} = \frac{(-1)^n}{2^n} .$$

Thus, if n is even, the right side is positive; in fact for $n = 2r$

$$t_{2r} - t_{2r-1} = \frac{1}{2^{2r}} ,$$

that is

$$U_r - L_r = \frac{1}{2^{2r}} .$$

This shows Property 4.

Furthermore,

$$t_{2r} - t_{2r-2} = \frac{(-1)^{2r}}{2^{2r}} + \frac{(-1)^{2r-1}}{2^{2r-1}}$$

$$= \frac{1}{2^{2r}} - \frac{1}{2^{2r-1}} = \frac{(1 - 2)}{2^{2r}}$$

which is negative. This shows that t_{2r} is less than t_{2r-2}, that is, U_r is less that U_{r-1}. Similarly it may be shown that L_r is greater than L_{r-1}.

To find the number represented we use the same trick as that as in Section 6.5 and write

$$t_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots + \frac{(-1)^{n-1}}{2^{n-1}} + \frac{(-1)^n}{2^n}$$

$$2t_n = 2 - 1 + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{(-1)^n}{2^{n-1}} .$$

Notice that $(-1)^n + (-1)^{n-1} = (-1 + 1)(-1)^{n-1} = 0$ and add the two expressions. Then

$$t_n + 2t_n = 2 + 0 + 0 + 0 + \dots + 0 + \frac{(-1)^n}{2^n} ,$$

thus

$$3t_n = 2 + \frac{(-1)^n}{2^n} .$$

The second term on the right becomes smaller and smaller as n becomes larger. Hence as n becomes larger and larger, $3t_n$ becomes closer and closer to 2. This means that the number represented by the two sequences must be $\frac{2}{3}$. Calculating some of the terms will show that this is reasonable.

Section 9.2

1. c) $x = \frac{2}{3}$

   d) $x = 2$ or -2.

3. If the given equation were satisfied by an integer x, the method of the previous exercise would work. That is, we note that the number 5 can be expressed as a product of two integers in just four ways. We could try them all as follows:

   a) $5 = 5 \cdot 1$, $x - 2 = 5$ and $x - 4 = 1$

   b) $5 = 1 \cdot 5$, $x - 2 = 1$ and $x - 4 = 5$

   c) $5 = (-5)(-1)$, $x - 2 = -5$ and $x - 4 = -1$

   d) $5 = (-1)(-5)$, $x - 2 = -1$ and $x - 4 = -5$

   In case a) we would have $x = 7$ and $x = 5$ which is impossible. Similarly the other three will be seen to be impossible. All we have shown by this is that there is no integer value of x which satisfies the equation.

The sequence of steps in the exercise below shows that the solutions of the given equation are in fact irrational numbers.

Section 9.3

1. d) To simplify this inequality we want to multiply both sides by the product $(x - 5)(x + 6)$. But the resulting direction of the inequality depends on the sign of this product. Now, the product will be positive unless one member is negative and the other positive. But if $x - 5$ is positive, so is $x + 6$. Hence the only way the product can be negative is for $x - 5$ to be negative and $x + 6$ to be positive, that is, x is between -6 and 5. So we have two cases:

I. If x is between -6 and 5, then the product is negative and the given inequality is equivalent to:

$$2(x + 6) > 7(x - 5),$$

$$2x + 12 > 7x - 35$$

$$47 > 5x$$

$$\frac{47}{5} > x\ .$$

But if x is between -6 and 5 it is less than $\frac{47}{5}$. Hence for this case the inequality holds.

II. If x is not between -6 and 5, the inequality is reversed in the above sequence of steps and the final inequality is

$$\frac{47}{5} < x.$$

Here if x is greater than $\frac{47}{5}$ it is not between -6 and 5 and hence for all such x the inequality holds.

Combining the two results we have as the solution of the given inequality:

$$x > \frac{47}{5}\ \underline{\text{or}}\ \ x\ \text{ between -6 and 5.}$$

3. a) will be a line segment with one end-point omitted.

   b) is two half lines, or two rays with end-points omitted.

   c) is the whole line.

   d) no points at all.

Section 9.5

1. Since $ax + by = 1$ always has a solution in integers x and y when a and b are integers with g.c.f. 1, it follows that on the graph of this equation, there will always be lattice points, that is, points both of whose coordinates are integers.

Section 9.7

1. a) Here there are no solutions because the second equation is equivalent to $2x + 3y = \frac{15}{2}$ which cannot hold if $2x + 3y = 7$ as in the first equation.

2. For a) and b) there will be a V-shaped region of the plane. In part c) there will be no points satisfying the two conditions. In part d) it will be the portion of the plane between the two parallel lines.

3. The solutions are the pairs $(3,0)$ and $(5,4)$ together with those obtained by changing any or all of the signs.

5. Suppose $x + y = a$ and $x - y = b$. Then $2x = a + b$ and $2y = a - b$. This means that there will be a solution of the given pair of equations only if both $a + b$ and $a - b$ are even. Now if one is even, then the other is, because their difference is $2b$ which is even. Thus there is a solution of

$$(x + y)(x - y) = ab$$

if and only if $a + b$ is even, that is, a and b are both even or both odd. Thus if c is 64, then the product $1 \cdot 64$ will give no solutions but the product $2 \cdot 32$ will. Hence if c is a multiple of 4 we can make both factors even, if c is odd both factors must be odd. The difficulty arises when c is neither a multiple of 4 nor odd, that is, c is 2 more than a multiple of 4. In that case there will be no solutions since in any representation as a product of two integers one factor must be odd and the other even.

Section 9.8

1. One must drain $\frac{10}{7}$ of a gallon and replace by pure antifreeze.

2. They should buy 1.8 pounds of beef and .4 of pork.

3. Yes.

4. Referring to the solution of the problem we have now the equations

$$x + y = 5$$

$$rx + sy = 5t.$$

To solve for y we multiply the first equation by r and subtract the second from it. This gives $(r - s)y = 5(r - t)$ or $y = 5\,\frac{(r - t)}{(r - s)}$. To solve for x we may multiply the first equation by s and subtract this from the second equation to get $(r - s)x = 5(t - s)$ or $x = 5\frac{(t - s)}{(r - s)}$. Notice that to make sense x and y must be positive. This will happen if $r < t < s$ or $s < t < r$, in other words if t is between r and s.

INDEX

abelian group, 92, 125, 207
absolute value, 124, 192
addition, 88
algebraically complete number
    system, 191
amount, 113
approximation, 107
archimedian property, 158
arithmetic properties, 49
associative property
  for addition, 30
  for multiplication, 33
axis
  of pure imaginaries, 192
  of reals, 191

"Bertrand's Postulate," 71
betweenness, 137, 25

cancellation property, 126
  for addition, 40
  for inequality, 134
  for multiplication, 41
canonical form, 81
cartesian product, 101, 182
categorical sets, 132
classification, 82
closure property
  for addition, 28
  for multiplication, 33
commutative property
  of addition, 29
  of multiplication, 33
completeness postulate, 146
complex number, 187
  pure imaginary part of, 188
  real part of, 188
composite numbers, 52
conditional equations, 152
countable sets, 183
counting numbers, 23

decimal expansions, 110
  convergence of, 111
  of a real number, 147
decimal system, 10
decimals, 104, 111
De Moivre's Theorem, 197
density, 95, 96, 27
disjoint sets, 180, 132
distributive property, 5
  for whole numbers, 34

divisibility, 50
  test for by 9, 64
division, 85
divisor, 50
duodecimal system, 12

elements, 179
empty set, 180
equations
  equivalent, 152
  equivalent pairs of, 168
equivalence relation, 82
equivalent fractions, 78, 79
euclidean algorithm, 56
even number, 51

factor, 50
field, 131
functions, 164, 162
fundamental algebraic properties, 46
Fundamental Theorem of Arithmetic, 53

greatest common factor, 55
group under addition, 126, 207
Guatemala, 21

identity, 152
identity element
  for addition, 29
  for multiplication, 34
imaginary number, 188
inequalities, 43, 132
  cancellation property for, 134
  definition of, 132
  solution of, 156
integers, 118
  cancellation property of, 126
  multiplication of, 122
  negative, 118
  positive, 118
  subtraction of, 121
integral domain, 126
interest, 113
  compound interest formula, 114
  continuous interest, 115
intersection, 180
inverse operations, 37

lattice point, 98
least common multiple, 60
lemma, 156
lowest terms, 79

modular arithmetic, 205
  modulo seven, 208
  modulo twelve, 205
multiple, 50
multiplication, 84
multiplicative group, 92
multiplicative inverses, 90

natural numbers, 23
negative integers, 118, 127
  multiplication of, [illegible]
non-negative rational numbers, 90
null set, 180
number pairs, 127, 131
number systems, 9
number zero, 27
numeral systems,
  the base of, 11
  base five, 16
  base two, 15
numeral zero, 27, 11

odd number, 1
one-to-one correspondence, 6, 24
opposite number, 11
origin, 99

pairs
  ordered, 127, 98
  of linear equations, 168
percentage, 113
periodic decimals, 111
positive integers, 118
prime numbers, 52
principal, 113
properties
  of equivalent relations, 52
  of inequalities, 43
  of order, 25
  of rational number system, 190

rational numbers, 77
  abelian group of, 191
  addition of, 88
  division of, [illegible]
  multiplication of, 84
  non-negative, 117, 118
  subtraction of, 94
ratios, 104
real number
  definition of, 142
reciprocals, 55
relations, 182
repeating decimals, 111
repetend, 111

scientific notation, 107
sets
  cartesian product of, 182
  disjoint, 180
  elements of, 179
  empty, 180
  intersection of, 180
  ordered, 25
  relationships between, 179
  union of, 181
similarity, 199
simplest form, 79, 81
slope, 199
solution set, 152
subset, 179

transitive property, 31, 35, 88
triangle inequality, 138

union of sets, 28, 181
unit
  in decimal system, 10
  in other systems, [illegible]

well-defined property
  for addition, 30
  for multiplication, 33
whole numbers, 23
  fundamental algebraic properties of, 46
  properties of inequality of, 46